1. The general state of a spin half particle with spin component \( S_n = \mathbf{S} \cdot \hat{n} = \frac{1}{2} \hbar \) can be shown to be given by

\[
|S_n = \frac{1}{2} \hbar\rangle = \cos\left(\frac{1}{2} \theta\right)|S_z = \frac{1}{2} \hbar\rangle + e^{i\phi} \sin\left(\frac{1}{2} \theta\right)|S_z = -\frac{1}{2} \hbar\rangle
\]

where \( \hat{n} \) is a unit vector \( \hat{n} = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k} \), with \( \theta \) and \( \phi \) the usual angles for spherical polar coordinates.

(a) Determine the expression for the states \( |S_x = \frac{1}{2} \hbar\rangle \) and \( |S_y = \frac{1}{2} \hbar\rangle \).

(b) Suppose that a measurement of \( S_z \) is carried out on a particle in the state \( |S_n = \frac{1}{2} \hbar\rangle \). What is the probability that the measurement yields each of \( \pm \frac{1}{2} \hbar \)?

(c) Determine the expression for the state for which \( S_n = -\frac{1}{2} \hbar \).

(d) Calculate the inner products \( \langle S_n = -\frac{1}{2} \hbar|S_n = \frac{1}{2} \hbar\rangle \), \( \langle S_n = \frac{1}{2} \hbar|S_n = \frac{1}{2} \hbar\rangle \) and \( \langle S_n = -\frac{1}{2} \hbar|S_n = -\frac{1}{2} \hbar\rangle \).

SOLUTION

(a) For the state \( |S_x = \frac{1}{2} \hbar\rangle \), the unit vector \( \hat{n} \) must be pointing in the direction of the X axis, i.e. \( \theta = \pi/2 \), \( \phi = 0 \), so that

\[
|S_x = \frac{1}{2} \hbar\rangle = \frac{1}{\sqrt{2}}\left[|S_z = \frac{1}{2} \hbar\rangle + |S_z = -\frac{1}{2} \hbar\rangle\right]
\]

For the state \( |S_y = \frac{1}{2} \hbar\rangle \), the unit vector \( \hat{n} \) must be pointed in the direction of the Y axis, i.e. \( \theta = \pi/2 \) and \( \phi = \pi/2 \). Thus

\[
|S_y = \frac{1}{2} \hbar\rangle = \frac{1}{\sqrt{2}}\left[|S_z = \frac{1}{2} \hbar\rangle + i|S_z = -\frac{1}{2} \hbar\rangle\right]
\]

(b) The probabilities will be given by \( |\langle S_z = \pm \frac{1}{2} \hbar|S_n = \frac{1}{2} \hbar\rangle|^2 \). These probabilities will be

\[
|\langle S_z = \frac{1}{2} \hbar|S_n = \frac{1}{2} \hbar\rangle|^2 = \cos^2\left(\frac{1}{2} \theta\right)
\]

\[
|\langle S_z = -\frac{1}{2} \hbar|S_n = \frac{1}{2} \hbar\rangle|^2 = |e^{i\phi} \sin\left(\frac{1}{2} \theta\right)|^2 = \sin^2\left(\frac{1}{2} \theta\right).
\]

(c) The state for which \( S_n = -\frac{1}{2} \hbar \) is the state for which \( \mathbf{S} \cdot \hat{n} = -\frac{1}{2} \hbar \), i.e. \( \mathbf{S} \cdot (-\hat{n}) = \frac{1}{2} \hbar \). Thus the required result can be obtained by making the replacement \( \hat{n} \to -\hat{n} \), which is achieved through the replacements \( \theta \to \pi - \theta \) and \( \phi \to \phi + \pi \). Thus, the required expression is

\[
|S_n = -\frac{1}{2} \hbar\rangle = \cos\left(\frac{1}{2} \left(\pi - \theta\right)\right)|S_z = \frac{1}{2} \hbar\rangle + e^{i(\phi+\pi)} \sin\left(\frac{1}{2} \left(\pi - \theta\right)\right)|S_z = -\frac{1}{2} \hbar\rangle
\]

\[
= \sin\left(\frac{1}{2} \theta\right)|S_z = \frac{1}{2} \hbar\rangle - e^{i\phi} \cos\left(\frac{1}{2} \theta\right)|S_z = -\frac{1}{2} \hbar\rangle
\]
(d) The required inner products are, using the orthonormality of the basis states 
$|S_z = \pm \frac{1}{2}\hbar|$
\[
\langle S_n = -\frac{1}{2}\hbar | S_n = \frac{1}{2}\hbar \rangle = \left[ \sin \left( \frac{1}{2} \theta \right) \langle S_z = \frac{1}{2}\hbar | - e^{-i\phi} \cos \left( \frac{1}{2} \theta \right) \langle S_z = -\frac{1}{2}\hbar \rangle \right] \\
\times \left[ \cos \left( \frac{1}{2} \theta \right) | S_z = \frac{1}{2}\hbar \rangle + e^{i\phi} \sin \left( \frac{1}{2} \theta \right) | S_z = -\frac{1}{2}\hbar \rangle \right] \\
= \sin \left( \frac{1}{2} \theta \right) \cos \left( \frac{1}{2} \theta \right) - \cos \left( \frac{1}{2} \theta \right) \sin \left( \frac{1}{2} \theta \right) = 0
\]
\[
\langle S_n = \frac{1}{2}\hbar | S_n = \frac{1}{2}\hbar \rangle = \left[ \cos \left( \frac{1}{2} \theta \right) \langle S_z = \frac{1}{2}\hbar | + e^{-i\phi} \sin \left( \frac{1}{2} \theta \right) \langle S_z = -\frac{1}{2}\hbar \rangle \right] \\
\times \left[ \cos \left( \frac{1}{2} \theta \right) | S_z = \frac{1}{2}\hbar \rangle + e^{i\phi} \sin \left( \frac{1}{2} \theta \right) | S_z = -\frac{1}{2}\hbar \rangle \right] \\
= \cos^2 \left( \frac{1}{2} \theta \right) + \sin^2 \left( \frac{1}{2} \theta \right) = 1.
\]
\[
\langle S_n = -\frac{1}{2}\hbar | S_n = -\frac{1}{2}\hbar \rangle = \left[ \sin \left( \frac{1}{2} \theta \right) \langle S_z = \frac{1}{2}\hbar | - e^{-i\phi} \cos \left( \frac{1}{2} \theta \right) \langle S_z = -\frac{1}{2}\hbar \rangle \right] \\
\times \left[ \sin \left( \frac{1}{2} \theta \right) | S_z = \frac{1}{2}\hbar \rangle - e^{i\phi} \cos \left( \frac{1}{2} \theta \right) | S_z = -\frac{1}{2}\hbar \rangle \right] \\
= \sin^2 \left( \frac{1}{2} \theta \right) + \cos^2 \left( \frac{1}{2} \theta \right) = 1.
\]
2. A beam of spin half particles is sent through a series of three Stern-Gerlach devices, as illustrated in the figure below. The first device transmits particles with $S_z = \frac{1}{2} \hbar$ and filters out particles with $S_z = -\frac{1}{2} \hbar$ (i.e. this beam is blocked). In the second device, the magnetic field is oriented in the direction $\hat{n}$, which lies in the XZ plane and makes an angle $\theta$ with the Z axis. This device transmits particles with $S_n = S \cdot \hat{n} = \frac{1}{2} \hbar$ and filters out particles with $S_n = -\frac{1}{2} \hbar$. A last device transmits particles with $S_z = -\frac{1}{2} \hbar$ and filters out particles with $S_z = \frac{1}{2} \hbar$.

(a) What fraction of the particles transmitted through the first Stern-Gerlach device will survive the third measurement? [You will need the probability of transmission through the second Stern-Gerlach apparatus set at an angle $\theta$ obtained in Question 1.]

(b) How must the angle $\theta$ of the second device be oriented so as to maximize the number of particles that are transmitted by the final device? What fraction of the particles survive the third measurement for this value of $\theta$?

(c) What fraction of the particles survive the last measurement if the second Stern-Gerlach device is simply removed from the experiment?

SOLUTION

(a) As the relative angle of orientation of the two magnetic fields between the first and second Stern-Gerlach devices is $\theta$, the probability of an atom emerging from the second device with $S_n = \frac{1}{2} \hbar$ will be $\cos^2(\frac{1}{2} \theta)$. The relative angle of the magnetic fields between the second and the third devices is $-\theta$ so that the probability of an atom emerging from the second device with $S_z = \frac{1}{2} \hbar$ is $\cos^2(-\frac{1}{2} \theta) = \cos^2(\frac{1}{2} \theta)$ and the probability of emerging with $S_z = -\frac{1}{2} \hbar$ is $\cos^2(-\frac{1}{2} (\pi - \theta)) = \sin^2(\frac{1}{2} \theta)$. Consequently the probability of an atom emerging in the final $S_z = -\frac{1}{2} \hbar$ beam is $\cos^2(\frac{1}{2} \theta) \sin^2(\frac{1}{2} \theta) = \frac{1}{4} \sin^2 \theta$.

(b) The maximum value that $\sin^2 \theta$ can have is unity. This will occur when $\theta = \pi/2$, i.e. when the second device has its magnetic field oriented in the $x$ direction. The maximum fraction of atoms that will make it through the series of Stern-Gerlach devices is $\frac{1}{4}$. Half will be lost in passing through the second device, and of those that make it through, another half will be lost when passing through the final device.

(c) If the intervening device is simply removed, then no atoms will emerge from the last device in the $S_z = -\frac{1}{2} \hbar$ beam as all atoms entering the device will have $S_z = \frac{1}{2} \hbar$. 

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3. A neutral molecule consisting of 3 identical atoms sitting at the vertices \( A, B, C \) of an equilateral triangle is ionized by the addition of an electron which can be attached to any one of the three atoms. Let the state of the system in which the electron is attached to atom \( A \) be \( |A\rangle \) etc.

(a) Show that the states

\[
|1\rangle = \frac{1}{\sqrt{3}} \left[ |A\rangle + e^{-2i\pi/3} |B\rangle + e^{2i\pi/3} |C\rangle \right]
\]

\[
|2\rangle = \frac{1}{\sqrt{3}} \left[ |A\rangle + |B\rangle + |C\rangle \right]
\]

are orthonormal.

(b) What is the third state required to make these states a complete orthonormal set of basis states?

SOLUTION

(a) First check that the states are normalized:

\[
\langle 1|1\rangle = \frac{1}{3} \left[ \langle 1| + e^{2i\pi/3} \langle 2| + e^{-2i\pi/3} \langle 3| \right] \left[ |1\rangle + e^{-2i\pi/3} |2\rangle + e^{2i\pi/3} |3\rangle \right] = \frac{1}{3} (1 + 1 + 1) = 1
\]

\[
\langle 2|2\rangle = \frac{1}{3} \left[ \langle 1| + \langle 2| + \langle 3| \right] \left[ |1\rangle + |2\rangle + |3\rangle \right] = \frac{1}{3} (1 + 1 + 1) = 1.
\]

Now for orthogonality:

\[
\langle 2|1\rangle = \frac{1}{3} \left[ \langle 1| + e^{2i\pi/3} \langle 2| + e^{-2i\pi/3} \langle 3| \right] \left[ |1\rangle + e^{2i\pi/3} |2\rangle + e^{-2i\pi/3} |3\rangle \right] = \frac{1}{3} \left[ 1 + e^{2i\pi/3} + e^{-2i\pi/3} \right] = \frac{1}{3} \left[ 1 + 2 \cos(2\pi/3) \right] = 0 \text{ since } \cos(2\pi/3) = -\frac{1}{2}.
\]

(b) Suppose we let the third state be given by

\[
|3\rangle = a|A\rangle + b|B\rangle + c|C\rangle.
\]

We then have, by the orthogonality of \( |3\rangle \) with \( |1\rangle \) and \( |2\rangle \):

\[
\langle 1|3\rangle = a + e^{2i\pi/3} b + e^{-2i\pi/3} c = 0 \quad (1)
\]

\[
\langle 2|3\rangle = a + b + c = 0. \quad (2)
\]

Eliminating \( a \) from this pair of equations gives

\[
(e^{2i\pi/3} - 1)b + (e^{-2i\pi/3} - 1)c = 0
\]

which can be written

\[
e^{i\pi/3}(e^{i\pi/3} - e^{-i\pi/3})b + e^{-i\pi/3}(e^{-i\pi/3} - e^{i\pi/3})c = 0
\]
Cancelling the common factor then gives
\[ e^{i\pi/3}b - e^{-i\pi/3}c = 0 \]
or
\[ b = e^{-2i\pi/3}c. \]  \hspace{1cm} (3)

If we now multiply Eq. (1) by \( \exp(2i\pi/3) \) and use the fact that \( \exp(4i\pi/3) = \exp(-2i\pi/3) \) we get
\[ e^{2i\pi/3}a + e^{-2i\pi/3}b + c = 0 \]
which becomes, on reordering the terms
\[ c + e^{2i\pi/3}a + e^{-2i\pi/3}b = 0 \]
\[ c + b + a = 0 \]
which is of the same form as the original pair of equations with \( b \rightarrow a, a \rightarrow c \) and \( c \rightarrow b \) so that the solution will be
\[ a = e^{-2i\pi/3}b. \]  \hspace{1cm} (4)

Combining Eq. (3) and Eq. (4) then gives
\[ b = e^{2i\pi/3}a \quad \text{and} \quad c = e^{-2i\pi/3}a \]
Thus, overall, we can write
\[ |3\rangle = a[|A\rangle + e^{2i\pi/3}|B\rangle + e^{-2i\pi/3}|C\rangle] \]
Checking the normalization to determine \( a \) then shows that the final normalized state is
\[ |3\rangle = \frac{1}{\sqrt{3}}[|A\rangle + e^{2i\pi/3}|B\rangle + e^{-2i\pi/3}|C\rangle]. \]

4. A cavity which supports a single mode of oscillation of the electromagnetic field can have states in which there are \( n = 0, 1, 2, \ldots \) photons present, from which we can construct the basis vectors \( \{|n\rangle; n = 0, 1, 2, \ldots \} \). For the following two states,
\[ |\psi\rangle = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} |n\rangle \quad \text{and} \quad |\phi\rangle = \sum_{n=1}^{\infty} \frac{1}{n} |n\rangle \]
determine whether or not the states are normalizable, and if so, write down the expression for the normalized state. Which of the states represents a possible physical state of the system?

**SOLUTION**

The inner product \( \langle \psi | \psi \rangle \) is given by
\[ \langle \psi | \psi \rangle = \sum_{n=1}^{\infty} \frac{1}{n} \]
which is a divergent series, and hence the state vector \( |\psi\rangle \) cannot be normalized to unity. No probability interpretation can be applied in this case, and hence this vector does not represent a possible physical state of the system.
The inner product $\langle \phi | \phi \rangle$ in contrast is
\[
\langle \phi | \phi \rangle = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]
which is a convergent series – it converges to $\pi^2/6$ – so the state vector $|\phi\rangle$ can be normalized to unity:
\[
|\bar{\phi}\rangle = \frac{\sqrt{6}}{\pi} |\phi\rangle,
\]
and hence can represent a physical state of the system.

5. With respect to a pair of orthonormal vectors $|1\rangle$ and $|2\rangle$ that span the state space $\mathcal{H}$ of a certain system, the Hermitean operator $\hat{Q}$ is defined by its action on these base states as follows:
\[
\hat{Q}|1\rangle = 2|1\rangle - 2i|2\rangle \quad \hat{Q}|2\rangle = 2i|1\rangle - |2\rangle.
\]
(a) What is the matrix representation of $\hat{Q}$ in the $\{|1\rangle, |2\rangle\}$ basis?
(b) Show that the states
\[
|q_1\rangle = \frac{1}{\sqrt{5}} (|1\rangle + 2i|2\rangle) \quad |q_2\rangle = \frac{1}{\sqrt{5}} (2|1\rangle - i|2\rangle)
\]
are eigenstates of $\hat{Q}$ and determine the associated eigenvalues.
(c) The operator $\hat{Q}$ above represents a certain physical observable $Q$ of a quantum system which is prepared in the state
\[
|\psi\rangle = \frac{1}{\sqrt{3}} |1\rangle + \frac{1}{\sqrt{3}} + i \sqrt{3} |2\rangle.
\]
(i) What are the possible results of a measurement of the observable $Q$?
(ii) What are the probabilities of obtaining each of the possible results?
(iii) What is the state of the system after the measurement is performed for each of the possible measurement outcomes?

SOLUTION

(a) The required matrix is
\[
\hat{Q} = \begin{pmatrix} 2 & 2i \\ -2i & -1 \end{pmatrix}
\]
(b) Letting $\hat{Q}$ act on $|q_1\rangle$ gives
\[
\hat{Q}|q_1\rangle = \frac{1}{\sqrt{5}} \hat{Q}( |1\rangle + 2i|2\rangle )
= \frac{1}{\sqrt{5}} (\hat{Q}|1\rangle + 2i\hat{Q}|2\rangle )
= \frac{1}{\sqrt{5}} (2|1\rangle - 2i|2\rangle - 4|1\rangle - 2i|2\rangle )
= \frac{1}{\sqrt{5}} (-2|1\rangle - 4i|2\rangle )
= -2|q_1\rangle.
\]
Thus, $|q_1\rangle$ is an eigenstate of $\hat{Q}$ with eigenvalue $-2$. Similarly, letting $\hat{Q}$ act on $|q_2\rangle$ gives

$$\hat{Q}|q_2\rangle = \frac{1}{\sqrt{5}}\hat{Q}(2|1\rangle - i|2\rangle)$$

$$= \frac{1}{\sqrt{5}}(2\hat{Q}|1\rangle - i\hat{Q}|2\rangle)$$

$$= \frac{1}{\sqrt{5}}(4|1\rangle - 4i|2\rangle + 2|1\rangle + i|2\rangle)$$

$$= \frac{1}{\sqrt{5}}(6|1\rangle - 3i|2\rangle)$$

$$= 3|q_2\rangle.$$ 

Thus, $|q_2\rangle$ is an eigenstate of $\hat{Q}$ with eigenvalue 3.

(c) (i) The possible results are the eigenvalues of $\hat{Q}$, that is -2 or 3.

(ii) The probability of obtaining the result $-2$ will be $|\langle q_1|\psi\rangle|^2$ and this is given by

$$|\langle q_1|\psi\rangle|^2 = \frac{1}{3}\left|\langle q_1|1\rangle + (1 + i)\langle q_1|2\rangle\right|^2$$

$$= \frac{1}{3}\left|\frac{1}{\sqrt{5}} - \frac{2i}{\sqrt{5}}(1 + i)\right|^2$$

$$= \frac{1}{15}|1 - 2i + 2|^2$$

$$= \frac{13}{15}.$$ 

Here we have made use of the expression for $|q_1\rangle$ in terms of the states $|1\rangle$ and $|2\rangle$ to calculate the inner products $\langle q_1|1\rangle$ and $\langle q_2|1\rangle$. The probability of obtaining the other result can be calculated in the same manner as above, or by simply subtracting this first result from unity. We will follow the longer route as a check on the calculations. Thus we require $|\langle q_2|\psi\rangle|^2$ which is given by

$$|\langle q_2|\psi\rangle|^2 = \frac{1}{3}\left|\langle q_2|1\rangle + (1 + i)\langle q_2|2\rangle\right|^2$$

$$= \frac{1}{3}\left|\frac{2}{\sqrt{5}} + \frac{i}{\sqrt{5}}(1 + i)\right|^2$$

$$= \frac{1}{15}|2 + i - 1|^2$$

$$= \frac{2}{15}.$$ 

(iii) If the result $q_1 = -2$ is obtained, then the state of the system immediately after the measurement is performed is $|q_1\rangle$, and if the result $q_2 = 3$ is obtained, then the state of the system immediately after the measurement is performed is $|q_2\rangle$. 

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