1. The general state of a spin half particle with spin component $S_n = S \cdot \hat{n} = \frac{1}{2} \hbar$ can be shown to be given by

$$|S_n = \frac{1}{2} \hbar\rangle = \cos(\frac{1}{2} \theta)|S_z = \frac{1}{2} \hbar\rangle + e^{i\phi} \sin(\frac{1}{2} \theta)|S_z = -\frac{1}{2} \hbar\rangle$$

where $\hat{n}$ is a unit vector $\hat{n} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$, with $\theta$ and $\phi$ the usual angles for spherical polar coordinates.

(a) Determine the expression for the the states $|S_x = \frac{1}{2} \hbar\rangle$ and $|S_y = \frac{1}{2} \hbar\rangle$.

(b) Suppose that a measurement of $S_z$ is carried out on a particle in the state $|S_n = \frac{1}{2} \hbar\rangle$. What is the probability that the measurement yields each of $\pm \frac{1}{2} \hbar$?

(c) Determine the expression for the state for which $S_n = -\frac{1}{2} \hbar$.

(d) Calculate the inner products $\langle S_n = -\frac{1}{2} \hbar | S_n = \frac{1}{2} \hbar \rangle$, $\langle S_n = \frac{1}{2} \hbar | S_n = \frac{1}{2} \hbar \rangle$ and $\langle S_n = -\frac{1}{2} \hbar | S_n = -\frac{1}{2} \hbar \rangle$.

2. A beam of spin half particles is sent through a series of three Stern-Gerlach apparatuses, as illustrated in the figure below. The first apparatus transmits particles with $S_z = \frac{1}{2} \hbar$ and filters out particles with $S_z = -\frac{1}{2} \hbar$ (i.e. this beam is blocked). In the second device, the magnetic field is oriented in the direction $\hat{n}$, which lies in the $XZ$ plane and makes an angle $\theta$ with the $Z$ axis. This apparatus transmits particles with $S_n = S \cdot \hat{n} = \frac{1}{2} \hbar$ and filters out particles with $S_n = -\frac{1}{2} \hbar$. A last apparatus transmits particles with $S_z = -\frac{1}{2} \hbar$ and filters out particles with $S_z = \frac{1}{2} \hbar$.

(a) What fraction of the particles transmitted through the first Stern-Gerlach apparatus (that is, the particles that are not blocked after exiting the first Stern-Gerlach apparatus) will survive the third measurement? [You will need the probability of transmission through the second Stern-Gerlach apparatus set at an angle $\theta$ obtained in Question 1.]

(b) How must the angle $\theta$ of the second apparatus be oriented so as to maximize the number of particles that are finally make it through the third apparatus? For this value of $\theta$ what fraction of the particles that make it through the first apparatus also survive the third measurement, i.e. are not blocked at the third apparatus?

(c) What fraction of the particles survive the last measurement if the second Stern-Gerlach apparatus is simply removed from the experiment?
3. A neutral molecule consisting of 3 identical atoms sitting at the vertices \( A, B, C \) of an equilateral triangle is ionized by the addition of an electron which can be attached to any one of the three atoms. Let the state of the system in which the electron is attached to atom \( A \) be \( |A \rangle \) etc.

(a) Show that the states

\[
|1\rangle = \frac{1}{\sqrt{3}} (|A\rangle + e^{-2i\pi/3}|B\rangle + e^{2i\pi/3}|C\rangle) \\
|2\rangle = \frac{1}{\sqrt{3}} (|A\rangle + |B\rangle + |C\rangle)
\]

are orthonormal.

(b) What is the third state required to make these states a complete orthonormal set of basis states?

4. A cavity which supports a single mode of oscillation of the electromagnetic field can have states in which there are \( n = 0, 1, 2, \ldots \) photons present, from which we can construct the basis vectors \( \{|n\rangle, n = 0, 1, 2, \ldots \} \). For the following two states

\[
|\psi\rangle = \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} |n\rangle \\
|\phi\rangle = \sum_{n=1}^{+\infty} \frac{1}{n} |n\rangle
\]

determine whether or not the states are normalizable, and if so, write down the expression for the normalized state. Which of the states represents a possible physical state of the system?

5. With respect to a pair of orthonormal vectors \( |1\rangle \) and \( |2\rangle \) that span the state space \( \mathcal{H} \) of a certain system, the Hermitean operator \( \hat{Q} \) is defined by its action on these base states as follows:

\[
\hat{Q}|1\rangle = 2|1\rangle - 2i|2\rangle \\
\hat{Q}|2\rangle = 2i|1\rangle - |2\rangle.
\]

(a) What is the matrix representation of \( \hat{Q} \) in the \( \{|1\rangle, |2\rangle\} \) basis?

(b) Show that the states

\[
|q_1\rangle = \frac{1}{\sqrt{3}} (|1\rangle + 2i|2\rangle) \\
|q_2\rangle = \frac{1}{\sqrt{3}} (2|1\rangle - i|2\rangle)
\]

are eigenstates of \( \hat{Q} \) and determine the associated eigenvalues.

(c) The operator \( \hat{Q} \) above represents a certain physical observable \( Q \) of a quantum system which is prepared in the state

\[
|\psi\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{1+i}{\sqrt{3}}|2\rangle.
\]

(i) What are the possible results of a measurement of the observable \( Q \)?

(ii) What are the probabilities of obtaining each of the possible results?

(iii) What is the state of the system after the measurement is performed for each of the possible measurement outcomes?