Sample Problems on Representations of Vectors and Operators for PHYS301

1. Consider two vectors

\[ |1\rangle = \frac{1}{\sqrt{2}}([-i] + |+\rangle) \quad \text{and} \quad |2\rangle = \frac{1}{\sqrt{2}}([-] + i|+) \]

where \(|\pm\rangle\) are the usual base vectors for a spin half system, and the operator \(\hat{A}\) defined by

\[ \hat{A}|\pm\rangle = \pm \frac{1}{2} i\hbar |\mp\rangle. \]

Note: expressed as column vectors in the \(|+, -\rangle\) representation, the basis states are

\[ |+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\mp\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

(a) Express the state vectors \(|1\rangle\) and \(|2\rangle\) as column vectors.
(b) Write down the corresponding bra vectors as row vectors.
(c) Calculate the inner products \(\langle 1|2\rangle\) and \(\langle 1|1\rangle\).
(d) Write this operator as a matrix in the \(|+, -\rangle\) representation.
(e) Calculate \(\hat{A}|1\rangle\) using the matrix representation of \(\hat{A}\) and the column representation of \(|1\rangle\).
(f) Calculate \(\langle 2|\hat{A}\) using the matrix representation of \(\hat{A}\) and the row representation of \(\langle 2|\).

SOLUTION

(a) The state vectors \(|1\rangle\) and \(|2\rangle\) are

\[ |1\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \text{and} \quad |2\rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}. \]

(b) The corresponding bra vectors are then

\[ \langle 1| \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \langle 2| \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}. \]

(c) Calculating the inner products using these representations gives

\[ \langle 1|2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{2} (-1 + 1) = 0, \]

and, similarly

\[ \langle 1|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{2} (1 + 1) = 1. \]
(d) Writing $\hat{A}$ out as a matrix:

$$\hat{A} = \begin{pmatrix} \langle + | \hat{A} | + \rangle & \langle + | \hat{A} | - \rangle \\ \langle - | \hat{A} | + \rangle & \langle - | \hat{A} | - \rangle \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix}.$$ 

The manner in which each element of the matrix is evaluated is illustrated here for $\langle - | \hat{A} | + \rangle$. Since $\hat{A}|+\rangle = \frac{1}{2}i\hbar|\pm\rangle$, then

$$\langle - | \hat{A} | + \rangle \equiv \langle - (\hat{A}|+\rangle) = \langle - | (\frac{1}{2}i\hbar|+\rangle) = \frac{1}{2}i\hbar\langle + | + \rangle = \frac{1}{2}i\hbar,$$

and the other elements follow in the same fashion.

(e) The action of $\hat{A}$ on $|1\rangle$, i.e. $\hat{A}|1\rangle$ can then be evaluated as follows:

$$\hat{A}|1\rangle = \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} = -\frac{1}{2}i\hbar \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2}h \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

i.e. $\hat{A}|1\rangle = \frac{1}{2}h|1\rangle$.

(f) Similarly, we can work out $\langle 2 | \hat{A}$:

$$\langle 2 | \hat{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \frac{1}{2}i\hbar \\ \frac{1}{2}i\hbar \end{pmatrix} = -\frac{1}{2}h \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

i.e. $\langle 2 | \hat{A} = -\frac{1}{2}h\langle 2 |$. 

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2
2. With respect to a pair of orthonormal vectors $|\varphi_1\rangle$ and $|\varphi_2\rangle$ that span the Hilbert space $\mathcal{H}$ of a certain system, the operator $\hat{A}$ is defined by its action on these base states as follows

$$\hat{A}|\varphi_1\rangle = 2|\varphi_1\rangle + 2i|\varphi_2\rangle$$
$$\hat{A}|\varphi_2\rangle = 2i|\varphi_1\rangle - |\varphi_2\rangle.$$ 

(a) What is the matrix representation of $\hat{A}$ in the $\{|\varphi_1\rangle, |\varphi_2\rangle\}$ basis?

(b) For the state $|\psi\rangle = (|\varphi_1\rangle + |\varphi_2\rangle)/\sqrt{2}$, determine $\hat{A}|\psi\rangle$ and $\langle \psi|\hat{A}$ by using the representations of the vectors and operators in the $\{|\varphi_1\rangle, |\varphi_2\rangle\}$ basis. Write your result in bra ket notation.

(c) Is $\langle \psi|\hat{A}$ the bra vector corresponding to $\hat{A}|\psi\rangle$?

SOLUTION

(a) The matrix representing $\hat{A}$ will be given by

$$\hat{A} = \begin{pmatrix}
\langle \varphi_1 | \hat{A} | \varphi_1 \rangle & \langle \varphi_1 | \hat{A} | \varphi_2 \rangle \\
\langle \varphi_2 | \hat{A} | \varphi_1 \rangle & \langle \varphi_2 | \hat{A} | \varphi_2 \rangle
\end{pmatrix}.$$ 

The various elements of this matrix are derived as follows:

$$\langle \varphi_1 | \hat{A} | \varphi_1 \rangle = \langle \varphi_1 | (\hat{A} | \varphi_1 \rangle) = \langle \varphi_1 | (2|\varphi_1\rangle + 2i|\varphi_2\rangle) = 2$$
$$\langle \varphi_1 | \hat{A} | \varphi_2 \rangle = \langle \varphi_1 | (\hat{A} | \varphi_2 \rangle) = \langle \varphi_1 | (2i|\varphi_1\rangle - |\varphi_2\rangle) = 2i$$
$$\langle \varphi_2 | \hat{A} | \varphi_1 \rangle = \langle \varphi_2 | (\hat{A} | \varphi_1 \rangle) = \langle \varphi_2 | (2|\varphi_1\rangle + 2i|\varphi_2\rangle) = 2i$$
$$\langle \varphi_2 | \hat{A} | \varphi_2 \rangle = \langle \varphi_2 | (\hat{A} | \varphi_2 \rangle) = \langle \varphi_2 | (2i|\varphi_1\rangle - |\varphi_2\rangle) = -1$$

and hence

$$\hat{A} = \begin{pmatrix}
2 & 2i \\
2i & -1
\end{pmatrix}.$$ 

(b) The column vector representing $|\psi\rangle$ will be

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that we have

$$\hat{A}|\psi\rangle = \begin{pmatrix}
2 & 2i \\
2i & -1
\end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 2 + 2i \\ -2i - 2i \end{pmatrix}$$

In bra ket notation this is

$$\hat{A}|\psi\rangle = \frac{1}{\sqrt{2}} [(2 + 2i)|\varphi_1\rangle - (1 - 2i)|\varphi_2\rangle].$$

Call this state vector $|\chi\rangle$. 

3
The bra corresponding to $|\psi\rangle$ is represented by the row vector

$$\langle \psi | = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and hence

$$\langle \psi | \hat{A} | \psi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2i \\ 2i & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 + 2i & -1 + 2i \end{pmatrix}$$

In bra ket notation this is:

$$\langle \psi | \hat{A} = \frac{1}{\sqrt{2}} [(2 + 2i)\langle \varphi_1 | - (1 - 2i)\langle \varphi_2 |]$$.  

(c) From above, we have $\hat{A}|\psi\rangle = |\chi\rangle$ where

$$|\chi\rangle = \frac{1}{\sqrt{2}} \left[ (2 + 2i)|\varphi_1\rangle - (1 - 2i)|\varphi_2\rangle \right].$$

The bra vector corresponding to this is then

$$\langle \chi | = \frac{1}{\sqrt{2}} \left[ (2 - 2i)\langle \varphi_1 | - (1 + 2i)\langle \varphi_2 | \right]$$

which is not the same as $\langle \psi | \hat{A}$.  

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4