Chapter 7

Probability Amplitudes

Evidence was presented in the preceding Chapter that under certain circumstances physical systems of quite disparate nature, but usually on the atomic scale, have the common properties of randomness and interference effects, that have no classical explanation. These general properties are the indicators, or the signatures, of the existence of basic physical laws that are not part of our normal everyday view of the way the world behaves, which immediately poses the problem of determining, and clearly stating, what these fundamental laws are. But there is in fact a second problem that has to be dealt with here. In order to state these basic laws, the question arises as to whether or not the concepts and mathematical language of classical physics is adequate for the task. We have seen that the mathematics of waves works fine for particles moving through space: from the wave function, which can be derived by solving the Schrödinger equation, information on the position and momentum, angular momentum, energy, and much else besides can be obtained. But we have also seen that the wave function cannot be used to describe spin. Apart from anything else, spin is a discrete variable, with a finite number of values, unlike the position of a particle which can vary in a continuous way. So what is needed is a mathematical language in terms of which these laws can be expressed, but which can assume outwardly different forms when applied to different physical systems. The two problems are not independent: the process of determining what the physical laws are guides us in the development of the mathematical language needed to express these laws. The outcome of all this, in another example of what has been termed 'the unreasonable effectiveness of mathematics in physics' is a mathematical language that was already well known to mathematicians in the early days of the quantum theory: the mathematics of linear vector spaces. It was a British theoretical physicist, Paul Dirac, who played a major role in formulating quantum mechanics in this way. He was inspired to do so, in part, because of the above-mentioned need for a theory that could cope with both particle spin and particle motion through space. What this work does, amongst other things, is bring into sharp focus the problem of defining what is meant by the state of a quantum system, and how a quantum state is to be represented mathematically. A further consequence of his work is a succinct notation which encapsulates both the physics and the mathematics, the Dirac bra-ket notation which is used, amongst other things, to represent the state of a quantum system. The first issue to be dealt with here is then that of coming to terms with what we mean by the state of a system, following which we look again at the two-slit experiment, which sets the scene for the development of the Dirac/Feynman version of quantum mechanics.

7.1 The State of a System

The notion of the state of a system is a central one in both classical and quantum physics, though it is often possible to live with only an intuitive idea of what it means. However, it proves to be important here to have the concept of the state of a system clearly defined. Ideally, specifying the state of a system would amount to gathering together all the information that it is possible to know about the system at any instant in time. The information should be enough so that, in principle, it would be possible to reconstruct the system at some other time and place and have the reconstructed system behave in exactly the same way as the original. From the point of view of physics, this information would take the form of numerical data that specifies, for instance for a single particle, its position and momentum at any instant in time. Of course, many systems when considered in all their detail are forbiddingly complex, but fortunately it is not always necessary to specify *everything* about a system. For instance, if we are interested in describing the orbit of the Earth around the sun, it would be sufficient to specify the state only in so far as the position and momentum of the centre of mass of the Earth. The fact that the Earth rotates, or that grass is green is not information that is required to deal with the orbital dynamics of the Earth. Knowing what is relevant and what is not is important in setting up good models of complex systems, but given that this has been decided upon for a given system, a good definition of the state of the system consistent with our intuitive notions is then to say that the state of a system is defined by specifying the maximum amount of data that can, in principle, be known simultaneously without mutual interference or contradiction about the system.

According to classical physics, it is possible in principle, if not always in practice, to determine *exactly* all the quantities needed to specify the state of a physical system. Thus, for a system consisting of a single particle, the state of the system is specified by giving the position and the momentum of the particle at the instant of interest. For a multi-particle system, the state could be specified by giving the positions and momenta of the individual particles, or else, if there are constraints of some kind, e.g. if the particles are organized into a single rigid body then it is probably sufficient to give the position and momentum of the centre of mass, the orientation of the body, and a few other quantities such as the linear and angular momentum. In practice, of course, there will always be uncertainty in our knowledge of these quantities, but this can be put down to inadequacies in our experimental technique, measuring apparatus, or the sheer size and complexity of the system under consideration.

When it comes to actually representing the state of a classical system, there is no way of doing this that has any significance beyond simply being a list of the values of all the physical parameters that can be determined for the system, though different ways of presenting this information offer different advantages in different circumstances. An example of one way of representing the information is, for a single particle, to plot the position and momentum of the particle as a point (x, p) in what is known as *phase space*. As a function of time, this point will trace out a path in phase space, this path then representing the evolution of the state of the system as a function of time.

The situation is somewhat different when quantum effects become important. It is not possible, even in principle, to specify the position *and* the momentum of a particle with total precision. The uncertainty principle tells us that we either have to compromise on the accuracy with which we specify each of the quantities, or else have to deal with the consequences of knowing, say, the position exactly, implying that the momentum is totally unknown, or vice versa. So what then are we to do with the notion of the state of a quantum particle? The situation is one of accepting the information that we have about the system, and taking that as specifying the state, which is consistent with the definition of state presented above, and is a reflection of what knowledge will really do possess about the system without mutual interference or contradiction. Thus, we can specify the position of a particle, but not its momentum, so we cannot represent the state of the system in the same way that we can for a classical system, as a point in phase space, for instance. Similarly we can only specify one component of the spin of a particle, say the x component of the spin of a particle, in which case we cannot specify its y or z component.

Here we will introduce a notation for the state of a quantum system which, for the present, is nothing more than a fancy way of writing down all the information that we can know about a quantum system, but which turns out to be very useful in describing the mathematical properties of quantum systems. The notation looks like this:

$$\left|\begin{matrix}\text{All the data concerning the system that can be known}\\ \text{without mutual interference or contradiction.}\end{matrix}\right\rangle \tag{7.1}$$

The symbol $|\;\rangle$ is known as a *ket*. Contained within the ket is a summary of the data specifying the state of a system, and hence a ket is also referred to as the state of the system. Thus, for instance, if we know the position of a particle is $x = 3$ cm with respect to some origin, then the state would be $|x = 3\text{ cm}\rangle$. If we know that a particle has a z component of spin equal to $\frac{1}{2}\hbar$, then the state would be $|S_z = \frac{1}{2}\hbar\rangle$. There would then be no such state as $|S_x = \frac{1}{2}\hbar, S_z = \frac{1}{2}\hbar\rangle$ since it is not possible to known both S_x and S_z simultaneously. At this stage there seems to be no point to enclosing the description of the state of a system within the symbol $|\;\rangle$, but, as will be seen later, quantum systems have the schizophrenic property of behaving as if they are *simultaneously* in a number of different states, and the way that this property of quantum systems is represented mathematically is by treating the states of a quantum system as if they are vectors in a way that we will be discussing later. Hence the above symbol is also known as a *ket vector* or *state vector*.

At times, when we want to be less specific, we would write $|x\rangle$ as being the state in which the x position of a particle is known, or $|p_x\rangle$ to be the state in which the x component of momentum is known, though the value in each case is unspecified. Finally, if we want to talk about a state without trying to spell out just what it is that is known precisely, i.e. an arbitrary state, we will write $|\psi\rangle$ or $|\phi\rangle$ or some such symbol. As a companion to this notation we will introduce a second way of writing down what this state is:

$$\left\langle\begin{matrix}\text{All the data concerning the system that can be known}\\ \text{without mutual interference or contradiction.}\end{matrix}\right|. \tag{7.2}$$

This symbol is known as a *bra* or *bra vector* and is equally well a way of representing the state of a quantum system. The distinction between a bra and a ket lies in the mathematics of quantum mechanics, which we will be dealing with later.

To see how this notation is motivated and employed, and how it acquires a mathematical meaning, we return to the two slit experiment.

7.2 The Two Slit Experiment Revisited

We now want to recast the two slit experiment in a fashion that leads to a new way of formulating quantum mechanics. The argument to be presented here is not meant to be rigorous, but more to suggest a way of thinking about quantum mechanics that can be made more precise, and very general in that it can be applied to any physical system. The experimental set up will be as before. Particles, which have all passed through exactly the same preparation procedure and hence are presumably all in the same state, are produced at a source S. These particles are then incident on a screen in which there are two narrow slits, 1 and 2, and beyond this screen is a further screen which the particles will ultimately strike, the point at which they hit this screen being registered in some way.

Since the particles are all prepared in the same way, they will presumably all be associated with the same wave function. The detailed form of the wave function will depend on many things, not the least of which is that the particles are all produced at the source S. So, to remind us of this, we will introduce the notation for the wave function

$$\Psi_S(x) = \text{probability amplitude of finding the particle at } x \text{ given that it originated at the source } S. \tag{7.3}$$

We will assume this wave function is unity at the source, i.e.

$$\Psi_S(S) = 1. \tag{7.4}$$

It is not important that we do this. We could choose otherwise, only to find that it cancels out at the end. Furthermore, we will not be concerning ourselves with the possible time dependence of the wave function here – in effect we are assuming some kind of steady state. This wave function will propagate through space and at the positions of the two slits will have values $\Psi_S(n)$, n =1, 2. These waves will then pass through these slits on their way towards the observation screen. The task then is to determine the amplitude of the waves at a point x on the observation screen due to waves that originated at each of the slits. To do that we will first of all suppose that, if the amplitude of the wave incident on slit 1 is unity, then the resulting wave amplitude at a position x on the screen is $\Psi_1(x)$. But since the amplitude of the wave at slit 1 is $\Psi_S(1)$ then it appears reasonable to suppose that we scale up the amplitude of the wave from slit 1 incident on the observation screen at x by the same factor, i.e. $\Psi_S(1)$, so that the amplitude of the wave incident on the screen at x will be $\Psi_S(1)\Psi_1(x)$. Of course, this is an assumption, but it is what is observed for, say, light waves passing through a slit – if the amplitude of a light wave incident on a slit is doubled, then the amplitude of the wave that emerges from the slit is also doubled. Light waves are *linear*. Probability amplitudes are hence also assumed to be linear.

In a similar way, the amplitude at x due to waves from slit 2 will be $\Psi_S(2)\Psi_2(x)$. Consequently, the total amplitude of the wave at x will be

$$\Psi_S(x) = \Psi_S(1)\Psi_1(x) + \Psi_S(2)\Psi_2(x) \tag{7.5}$$

where $\Psi_S(x)$ is the amplitude of waves at x that originated from the source S. This then is the probability amplitude of observing a particle at x given that it originated from the source S, i.e. by the Born interpretation Eq. (4.15), $|\Psi_S(x)|^2dx$ is the probability density of observing a particle in the region x to $x + dx$. This result is the same as the one presented in Eq. (4.14), but here we have expressed this total probability amplitude as a sum of two contributions of the form

$\Psi_S(n)\Psi_n(x) =$ Probability amplitude of observing the particle at slit n given that it originated from the source S.

$\times$ Probability amplitude of observing the particle at x given that it originated from slit n. (7.6)

$=$ Probability amplitude for the particle to pass from the source S to point x through slit n.

7.2.1 Sum of Amplitudes in Bra(c)ket Notation

It is at this point that we make use of the new notation we introduced in the preceding Section. Consider, for example, $\Psi_S(n)$, which we stated above as being the probability amplitude of observing the particle at slit n given that it originated from the source S. We could equivalently say that $\Psi_S(n)$ is the probability amplitude of observing the particle at the position of slit n, given that it was originally at the position of the source S, or even more precisely, that it is the probability amplitude of observing the particle to be in the state in which it is at the position of slit n, given that it was in a state in which it was at the position of the source S. If we use our notation from the previous Section to write

$$\begin{aligned} |S\rangle &\equiv \text{state of the particle when at the position of the source } S \\ |n\rangle &\equiv \text{state of the particle when at the position of slit } n \\ |x\rangle &\equiv \text{state of the particle when at the position } x \end{aligned} \tag{7.7}$$

we can then write, for instance,

$\Psi_S(n) =$ Probability amplitude of observing the particle in state $|n\rangle$ given that it was in state $|S\rangle$. (7.8)

This we will, finally, write as

$$\Psi_S(n) = \langle n||S\rangle = \langle n|S\rangle \tag{7.9}$$

i.e. we have written the final state as a bra, and where we have replaced the double vertical bar by a single bar. We can make similar replacements for the other probability amplitudes:

$$\Psi_S(x) \to \langle x|S\rangle; \quad \Psi_n(x) \to \langle x|n\rangle \tag{7.10}$$

i.e., for instance, $\langle x|S\rangle$ is the probability amplitude of finding the particle at x, i.e. in the state $|x\rangle$, given that it was initially at the source S, i.e. in the state $|S\rangle$. Recalling that a symbol such as $|S\rangle$ is also known as a ket, we can trace the origin of the names bra and ket to the fact that $\langle x|S\rangle$ can be thought of as a quantity enclosed in a pair of angled brackets, or 'bra(c)kets'. In terms of this new notation, the above result Eq. (7.5) becomes

$$\langle x|S\rangle = \langle x|1\rangle\langle 1|S\rangle + \langle x|2\rangle\langle 2|S\rangle. \tag{7.11}$$

Being able to write the probability amplitude in this way is a particularly important result as it leads directly to a new way of looking at the idea of the state of a physical system that lies at the heart of quantum mechanics.

7.2.2 Superposition of States for Two Slit Experiment

We can note that the expression Eq. (7.11) will hold true for all values of the variable x, i.e. it does not hold true for just one value of x. Because of this, we can do something rather radical and that is 'cancel' the $\langle x|$ to leave

$$|S\rangle = |1\rangle\langle 1|S\rangle + |2\rangle\langle 2|S\rangle \tag{7.12}$$

with the understanding that what we have created by doing so is a template into which we insert $\langle x|$ when we so desire, and thereby regain the expression Eq. (7.11). As a result of this step, we have apparently given a new meaning to the ket $|S\rangle$ as being more than just a way of summarizing all the information on the state of the system. The expression just obtained seems to suggest that there is a deeper mathematical and perhaps physical meaning that can be assigned to $|S\rangle$. For instance, we are free to manipulate Eq. (7.12) as we see fit. For instance we could solve for $|2\rangle$ to give

$$|2\rangle = \frac{|S\rangle - |1\rangle\langle 1|S\rangle}{\langle 2|S\rangle} \tag{7.13}$$

(recall that the $\langle\ldots|\ldots\rangle$ are all just complex numbers) and so on, and then put $\langle x|$ back in to give

$$\langle x|2\rangle = \frac{\langle x|S\rangle - \langle x|1\rangle\langle 1|S\rangle}{\langle 2|S\rangle}. \tag{7.14}$$

However, this development in the notation has more to offer than just this. Having created this new kind of expression, it is worthwhile to see whether or not it can be given any useful meaning. The new interpretation is a very potent one, and constitutes the central feature of quantum mechanics, the idea that a system, in some sense, can be simultaneously in a number of different physical states, otherwise known as a 'superposition of states'.

To see how this interpretation can be arrived at, we first of all note that since $\langle n|S\rangle$ is the probability amplitude of finding the particle at slit n, given that it was initially at the source S, and similarly for $\langle x|S\rangle$, $\langle x|n\rangle$, $\langle n|S\rangle$, then

$$\begin{aligned}
|\langle x|S\rangle|^2 &= \text{probability of finding the particle in state } |x\rangle \text{ given that it was in state } |S\rangle\\
|\langle x|n\rangle|^2 &= \text{probability of finding the particle in state } |x\rangle \text{ given that it was in state } |n\rangle\\
|\langle n|S\rangle|^2 &= \text{probability of finding the particle in state } |n\rangle \text{ given that it was in state } |S\rangle,
\end{aligned} \tag{7.15}$$

so that the coefficients $\langle 1|S\rangle$ and $\langle 2|S\rangle$ in Eq. (7.12) in some sense determine 'how much' of the state $|1\rangle$ is to be found in the initial state $|S\rangle$ and 'how much' of state $|2\rangle$ is to be found in $|S\rangle$. Put another way, the expression Eq. (7.12) indicates in a symbolic way the fact that when the particle is prepared in its initial state $|S\rangle$, there is built into this state the *potential* of the particle being found in the other states $|n\rangle$, with the chances of the particle being found in these other states being given by the coefficients $\langle n|S\rangle$. It is this sort of result that lies at the heart of quantum mechanics – the idea that a system in a certain state can behave as if it is in one or another of a number of other states, and to do so in a way that is probabilistic in nature. It is also out of this expression that the basic mathematical formalism of quantum mechanics is built, the idea being that the kets $|n\rangle$ can be considered to be, in some sense, vectors, much like the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ encountered in mechanics, and the coefficients $\langle n|S\rangle$ are the components of the total vector $|S\rangle$ along these two directions. It is this analogue that we will pursue in some detail in later Chapters. But for the present, we will show that the result Eq. (7.11) which we have 'derived' in the case of the two slit experiment can also be shown (and much more rigorously) to be an intrinsic part of the Stern-Gerlach experiment. In fact Eq. (7.11) is of a general form that is taken, ultimately, as a central postulate of quantum mechanics, one that is suggested by circumstantial evidence, though not proven. In effect, it is a law of nature.

7.3 The Stern-Gerlach Experiment Revisited

The aim here is to show that a result equivalent to the sum of amplitudes for different paths result obtained for the two slit experiment can also be shown to arise in the Stern-Gerlach experiment. However, unlike the previous result for the two slit arrangement, the result to be obtained here is more rigorously based and more useful in the long run as providing the insight necessary to generalize the result to any physical system.

Here will consider a fairly general kind of experimental arrangement illustrated in Fig. (7.1) in which there are two consecutive Stern-Gerlach devices, each of them with their magnetic fields oriented in an arbitrary direction, $\hat{\mathbf{n}}$ for the first, and $\hat{\mathbf{m}}$ for the second.

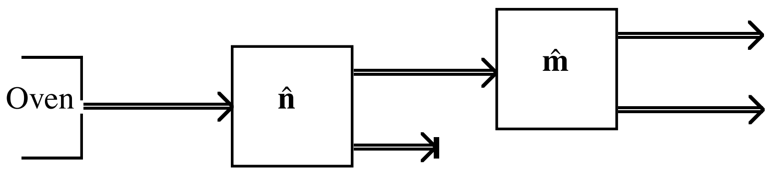

Figure 7.1: Same as Fig. (6.2) but with magnetic field of second Stern-Gerlach device oriented in $\hat{\mathbf{m}}$ direction.

Here, the first magnetic field makes an angle of θ_i with the z direction and the second an angle of θ_f with the z direction, see Fig. (7.2).

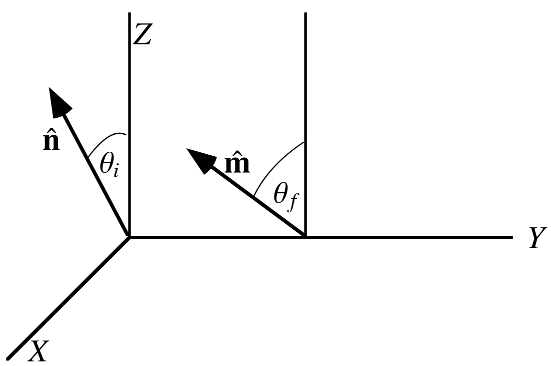

Figure 7.2: Orientation of the magnetic fields in the two Stern-Gerlach devices in Fig. (7.1)

This is actually not all that much different to the very first set-up given in Fig. (6.2) in which there is a pair of Stern-Gerlach devices, the first with the magnetic field in the $\hat{\mathbf{n}}$ direction, and the second with the magnetic field in in the z direction, so that the angle between the two fields is θ. We can get at the situation illustrated in Fig. (7.1) by simply imagining that the whole setup in Fig. (6.2) is rotated about the y axis through an angle of θ_f. This will make the second field (originally in the z direction in Fig. (6.2)) now point at an angle of θ_f, and the field in the first device now in the direction at an angle $\theta + \theta_f = \theta_i$ where $\theta = \theta_i - \theta_f$ is still the angle between the directions of the two magnetic fields. Consequently, the probability of an atom emerging in the beam for which $S_f = \mathbf{S} \cdot \hat{\mathbf{m}} = \frac{1}{2}\hbar$, given that it entered the last Stern-Gerlach device with $S_i = \mathbf{S} \cdot \hat{\mathbf{n}} = \frac{1}{2}\hbar$ will be (using $\cos\theta = \cos(-\theta)$)

$$P(S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar) = \cos^2(\tfrac{1}{2}\theta) = \cos^2[(\theta_f - \theta_i)/2]. \tag{7.16}$$

We now want to extract from this equation the mathematical statement that the atomic spins can pass through some intermediate states which we have the possibility of observing, i.e. by means of an device analogous to the set-up in Fig. (6.7). This amounts to being able to write Eq. (7.16) in the form

$$P(S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar) = |\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle|^2 \tag{7.17}$$

where $\langle S_f = \frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$ is, following the notation used in the previous Section, the probability amplitude of observing the $\hat{\mathbf{m}}$ component of the spin of an atom to be $\frac{1}{2}\hbar$ given that the $\hat{\mathbf{n}}$ component is known to be $\frac{1}{2}\hbar$, and moreover to be able to show that

$$\begin{aligned}\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle =& \langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle \\ &+ \langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle\end{aligned} \tag{7.18}$$

which is to be interpreted as meaning that the probability amplitude $\langle S_f = \frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$ of an atom being found in state $|S_f = \frac{1}{2}\hbar\rangle$ given that it was initially in the state $|S_i = \frac{1}{2}\hbar\rangle$ is the sum of the probability amplitudes of the atomic spin 'passing through', without observation, the two intermediate states $|S_I = \pm\frac{1}{2}\hbar\rangle$. A general form of this set-up would involve splitting then recombining the atomic beam in the direction of a unit vector $\hat{\mathbf{l}}$ aligned at some angle θ_I to the Z axis. Atoms will then emerge with spin components $S_I = \mathbf{S} \cdot \hat{\mathbf{l}}$ that, as usual, can have the two values $\pm\frac{1}{2}\hbar$. If $\theta_I = 0$, then we recover the situation in Fig. (6.7).

That this possibility is contained in the expression for the probability Eq. (7.16) can be seen by first writing

$$\cos[\tfrac{1}{2}(\theta_f - \theta_i)] = \cos[\tfrac{1}{2}(\theta_f - \theta_I) + \tfrac{1}{2}(\theta_I - \theta_i)]. \tag{7.19}$$

Using standard trigonometric formulae we get

$$\begin{aligned}&\cos[\tfrac{1}{2}(\theta_f - \theta_I) + \tfrac{1}{2}(\theta_I - \theta_i)] \\ &\qquad = \cos[\tfrac{1}{2}(\theta_f - \theta_I)]\cos[\tfrac{1}{2}(\theta_I - \theta_i)] - \sin[\tfrac{1}{2}(\theta_f - \theta_I)]\sin[\tfrac{1}{2}(\theta_I - \theta_i)].\end{aligned} \tag{7.20}$$

which becomes, on further use of simple trigonometry:

$$\sin[\tfrac{1}{2}(\theta_f - \theta_I)]\sin[\tfrac{1}{2}(\theta_I - \theta_i)] = -\cos[\tfrac{1}{2}(\theta_f - (\theta_I + \pi))]\cos[\tfrac{1}{2}((\theta_I + \pi) - \theta_i)]. \tag{7.21}$$

The probability $P(S_f = \frac{1}{2}\hbar|S_i = \frac{1}{2}\hbar)$ can therefore be written in the form

$$\begin{aligned}&P(S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar) = \cos^2[\tfrac{1}{2}(\theta_f - \theta_i)]\\&= \big(\cos[\tfrac{1}{2}(\theta_f - \theta_I)]\cos[\tfrac{1}{2}(\theta_I - \theta_i)] + \cos[\tfrac{1}{2}(\theta_f - (\theta_I + \pi))]\cos[\tfrac{1}{2}((\theta_I + \pi) - \theta_i)]\big)^2 \quad (7.22)\end{aligned}$$

This is already looking as if we have got what we were after. All that is necessary is to identify $\cos\frac{1}{2}(\theta_f - \theta_i)$ as the probability amplitude $\langle S_f = \frac{1}{2}\hbar|S_i = \frac{1}{2}\hbar\rangle$, with a similar identification being made for the factors $\cos[\frac{1}{2}(\theta_f - \theta_I)]$ and so on, this then leading to the required result given in Eq. (7.18). However, we are not quite there yet. In general, a probability amplitude is a complex quantity, and the probability is the modulus squared of a probability amplitude, so we can introduce complex phase factors into this expression provided we take the modulus square on the right hand side of Eq. (7.22) i.e. we write

$$\begin{aligned}P(S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar) =\ & |e^{i\varphi(\theta_f-\theta_i)}\cos[\tfrac{1}{2}(\theta_f - \theta_i)]|^2\\ =\ & \Big|e^{i\varphi(\theta_f-\theta_I)}\cos[\tfrac{1}{2}(\theta_f - \theta_I)]e^{i\varphi(\theta_I-\theta_i)}\cos[\tfrac{1}{2}(\theta_I - \theta_i)]\\ & + e^{i\varphi(\theta_f-\theta_I-\pi)}\cos[\tfrac{1}{2}(\theta_f - (\theta_I + \pi))]e^{i\varphi(\theta_I+\pi-\theta_i)}\cos[\tfrac{1}{2}((\theta_I + \pi) - \theta_i)]\Big|^2\end{aligned} \quad (7.23)$$

where we have associated the same form of phase factor with each of the cos factors appearing in Eq. (7.22). In essence, we are assuming that the probability amplitude $\langle S_f = \frac{1}{2}\hbar|S_i = \frac{1}{2}\hbar\rangle$ is a complex quantity – the cos factor multiplied by some as yet unspecified complex phase factor, but it remains to be seen what the possible form is for this phase factor. It can also be noted in passing that in introducing this phase factor, it has been recognized that the phase factors ought to be a function of angular differences only as it is only the difference in the angular settings of the Stern-Gerlach devices that can be of physical significance. We are now searching for some information about φ such that this last expression reduces to Eq. (7.22). So, expanding this expression gives

$$\begin{aligned}&P(S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar)\\&= \cos^2[\tfrac{1}{2}(\theta_f - \theta_I)]\cos^2[\tfrac{1}{2}(\theta_I - \theta_i)] + \cos^2[\tfrac{1}{2}(\theta_f - (\theta_I + \pi))]\cos^2[\tfrac{1}{2}((\theta_I + \pi) - \theta_i)]\\&\quad + 2\cos[\varphi(\theta_f - \theta_I) + \varphi(\theta_I - \theta_i) - \varphi(\theta_f - \theta_I - \pi) - \varphi(\theta_I + \pi - \theta_i)]\\&\quad \times \cos[\tfrac{1}{2}(\theta_f - \theta_I)]\cos[\tfrac{1}{2}(\theta_I - \theta_i)]\cos[\tfrac{1}{2}(\theta_f - (\theta_I + \pi))]\cos[\tfrac{1}{2}((\theta_I + \pi) - \theta_i)] \quad (7.24)\end{aligned}$$

In order for this expression to be the same as Eq. (7.22), we must have

$$\cos[\varphi(\theta_f - \theta_I) + \varphi(\theta_I - \theta_i) - \varphi(\theta_f - \theta_I - \pi) - \varphi(\theta_I + \pi - \theta_i)] = 1 \quad (7.25)$$

or

$$\varphi(\theta_f - \theta_I) + \varphi(\theta_I - \theta_i) - \varphi(\theta_f - \theta_I - \pi) - \varphi(\theta_I + \pi - \theta_i) = 2n\pi \quad (7.26)$$

where n is an integer. By rearranging terms, this becomes

$$\varphi(\theta_f - \theta_I) - \varphi(\theta_f - \theta_I - \pi) = 2n\pi + \varphi(\theta_I + \pi - \theta_i) - \varphi(\theta_I - \theta_i). \quad (7.27)$$

The left hand side of this equation is a function of θ_f while the right hand side is a function of θ_i, so each side must be independent of θ_i or θ_f, which can only be achieved if $\varphi(\theta)$ is a linear funcion of θ, i.e.

$$\varphi(\theta) = a\theta + b \quad (7.28)$$

which, on substituting into Eq. (7.27) shows that $n = 0$. The additive constant b represents a common phase shift which can be factored out so that the expression for the probability $P(S_f = \frac{1}{2}\hbar|S_i = \frac{1}{2}\hbar)$ becomes

$$P(S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar) = \Big|e^{ia(\theta_f-\theta_I)}\cos[\tfrac{1}{2}(\theta_f - \theta_I)]e^{ia(\theta_I-\theta_i)}\cos[\tfrac{1}{2}(\theta_I - \theta_i)] + e^{ia(\theta_f-\theta_I-\pi)}\cos[\tfrac{1}{2}(\theta_f - (\theta_I + \pi))]e^{ia(\theta_I+\pi-\theta_i)}\cos[\tfrac{1}{2}((\theta_I + \pi) - \theta_i)]\Big|^2 \quad (7.29)$$

The terms appearing here between the $|\ldots|$ can be regrouped to give

$$e^{ia(\theta_f-\theta_i)}\cos[\tfrac{1}{2}(\theta_f - \theta_i)] = e^{ia(\theta_f-\theta_I)}\cos[\tfrac{1}{2}(\theta_f - \theta_I)]e^{ia(\theta_I-\theta_i)}\cos[\tfrac{1}{2}(\theta_I - \theta_i)] + e^{ia(\theta_f-\theta_I-\pi)}\cos[\tfrac{1}{2}(\theta_f - (\theta_I + \pi))]e^{ia(\theta_I+\pi-\theta_i)}\cos[\tfrac{1}{2}((\theta_I + \pi) - \theta_i)] \quad (7.30)$$

with

$$P(S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar) = \Big|e^{ia(\theta_f-\theta_i)}\cos[\tfrac{1}{2}(\theta_f - \theta_i)]\Big|^2. \quad (7.31)$$

We can now note the consistent appearance in Eq. (7.30) and Eq. (7.31) of the quantity of the form

$$\psi(\theta) = e^{-ia\theta}\cos(\tfrac{1}{2}\theta). \quad (7.32)$$

Recalling that the probability $P(S_f = \frac{1}{2}\hbar|S_i = \frac{1}{2}\hbar)$ ought to be expressible as the modulus square of a probability amplitude, we can now tentatively identify, from Eq. (7.31), $\exp\{ia(\theta_f - \theta_i)\}\cos[(\theta_f - \theta_i)/2]$ as being, following the notation used in the previous Section, the probability amplitude $\langle S_f = \frac{1}{2}\hbar|S_i = \frac{1}{2}\hbar\rangle$ of observing the $\hat{\mathbf{m}}$ component of the spin of an atom to be $\frac{1}{2}\hbar$ given that the $\hat{\mathbf{n}}$ component is known to be $\frac{1}{2}\hbar$. Equivalently, this is the probability amplitude of observing the atom to be in the state $|S_f = \frac{1}{2}\hbar\rangle$ given that it is initially in the state $|S_i = \frac{1}{2}\hbar\rangle$, i.e.

$$\langle S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar\rangle = \psi(\theta_f - \theta_i) = e^{ia(\theta_f-\theta_i)}\cos[(\theta_f - \theta_i)/2], \quad (7.33)$$

We can now see that we can indeed write

$$P(S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar) = |\langle S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar\rangle|^2 \quad (7.34)$$

with

$$\langle S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar\rangle = \langle S_f = \tfrac{1}{2}\hbar|S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar\rangle + \langle S_f = \tfrac{1}{2}\hbar|S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar\rangle \quad (7.35)$$

as required. Thus, it is possible to define probability amplitudes for a spin half system in a way that yields this sum of products of probability amplitudes result in Eq. (7.18).

We can also pin down the phase factor a little further by noting that if we rotate a Stern-Gerlach apparatus through a full rotation of 2π, then we have reproduced the same physical set-up. Thus the probability amplitude $\psi(\theta)$ ought to be such that $\psi(\theta) = \psi(\theta + 2\pi)$, i.e.

$$e^{-ia\theta}\cos(\tfrac{1}{2}\theta) = e^{ia(\theta+2\pi)}\cos(\tfrac{1}{2}\theta + \pi) \quad (7.36)$$

and hence that

$$e^{2\pi ai} = -1. \quad (7.37)$$

which can be satisfied if we put $a = (n + \frac{1}{2})$ where n is an integer that can be chosen freely and is conventionally set equal to -1 so that

$$\langle S_f = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle = e^{i(\theta_2-\theta_1)/2}\cos[\tfrac{1}{2}(\theta_2 - \theta_1)] \tag{7.38}$$

Thus we have managed to show that the results for passing spin half systems through a Stern-Gerlach apparatus can be directly interpreted as being the consequence of the interference of probability amplitudes in a way analogous to the two slit experiment. Of course, if this could be done only for the two slit experiment and for spin half, then we would have to look on these two results as mere curiosities. But it turns out that these two instances are examples of a general principle that appears to apply to all physical systems, i.e. a new physical law.

The most familiar situation to which our result applies is that of Fig. (6.7) where the intermediate states are those for which $\theta_I = 0$, so that $S_I = S_z$, and we get

$$\begin{aligned}\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle =& \langle S_f = \tfrac{1}{2}\hbar | S_z = \tfrac{1}{2}\hbar\rangle\langle S_z = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle \\ &+ \langle S_f = \tfrac{1}{2}\hbar | S_z = -\tfrac{1}{2}\hbar\rangle\langle S_z = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle.\end{aligned} \tag{7.39}$$

However, the above general result shows that there is, in effect, an infinite number of intermediate states through which the atomic spin can pass.

The important feature of the result Eq. (7.18) is that it is expressed in terms of the probability amplitudes of the system passing through one or the other of the intermediate states for which $S_I = \mathbf{S}\cdot\hat{\mathbf{l}} = \pm\frac{1}{2}\hbar$, i.e. there appears here probability amplitudes that would be associated with the use of an intermediate Stern-Gerlach device to separate the atoms according to their S_I component of spin, *even though such an device need not ever appear in the actual experiment*. In fact, if we *do* insert the appropriate device with magnetic field in the $\hat{\mathbf{l}}$ direction to separate and then recombine the two beams, we find that *if we do not observe through which of the* $S_I = \pm\frac{1}{2}\hbar$ *beams the atoms pass*, then the probabilities for observing $S_f = \pm\frac{1}{2}\hbar$ for the recombined beam are exactly the same state as if the extra device were not present, i.e. the probability is given by Eq. (7.16), and is independent of the intermediate states that appear in Eq. (7.18). Put another way, having the intermediate device present, but not observing what it does, is the same as not having it there at all. On the other hand if we *do* observe which beam the atoms come through, then the observation probability is

$$\begin{aligned}P(S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar) = &|\langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle|^2 \\ &+ |\langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle|^2\end{aligned} \tag{7.40}$$

which does depend on the intermediate state that happens to be observed. These considerations show that built into the properties of the state of a quantum system is the *potential* for the system to be observed in *any* other state $|S_I = \frac{1}{2}\hbar\rangle$, for which the relevant probability amplitude $\langle S_I = \frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$ is non-zero. It is this property of the states of quantum systems that leads to the formulation of the state of a quantum system as a vector belonging to a complex vector space: the language of vectors makes it possible to express one vector (i.e. state) in terms of other vectors (i.e. states).

7.3.1 Superposition of States for Spin Half

Further development then follows what was done in Section 7.2. Without affecting the generality of the argument, we will specialize to the case illustrated in Fig. (7.3).

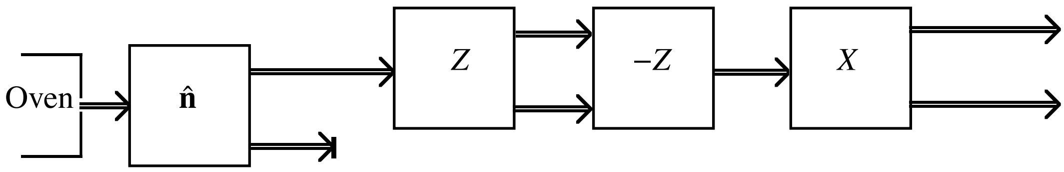

Figure 7.3: Atomic beam for which $S_i = \frac{1}{2}\hbar$ split into $S_z = \pm\frac{1}{2}\hbar$ beams and then recombined before passing through final Stern-Gerlach device with magnetic field in x direction.

This is the same setup as presented in Fig. (6.7) except that here the atom can emerge from the first Stern-Gerlach device in one or the other of two separate beams corresponding to the atomic spin component $S_i = \mathbf{S}\cdot\hat{\mathbf{n}} = \pm\frac{1}{2}\hbar$ where $\hat{\mathbf{n}}$ is some arbitrary orientation of the magnetic field (in the XZ plane) in the device. The atoms in one of the beams ($S_i = \frac{1}{2}\hbar$) is then selected and passed through a Stern-Gerlach device where the magnetic field further separates this beam according to its z component of spin.

Once again, it is important to see the analogy between this setup and the two slit interference experiment. The oven plus the first Stern-Gerlach device is the equivalent of the source of identically prepared particles in the two slit experiment. Here the atoms are all identically prepared to have $S_i = \frac{1}{2}\hbar$. The next two Stern-Gerlach devices are analogous to the two slits in that the atoms can, in principle, follow two different paths corresponding to $S_z = \pm\frac{1}{2}\hbar$ before they are recombined to emerge in one beam where their x component of spin is measured. The analogue is, of course, with a particle passing through one or the other of two slits before the position where it strikes the observation screen is observed. We can tell which path an atom follows (i.e. via the $S_z = \frac{1}{2}\hbar$ or the $S_z = -\frac{1}{2}\hbar$ beam) by monitoring which beam an atom emerges from after it passes through the first z oriented Stern-Gerlach device in much the same way that we can monitor which slit a particle passes through in the two slit experiment. Watching to see in which beam an atom finally emerges after passing through the last Stern-Gerlach device is then analogous to seeing where on the observation screen a particle lands after passing through the two slit device.

We want to analyze this experiment in a way that is analogous to the two slit experiment. Thus, we are assuming that all the atoms are prepared such that $S_i = \mathbf{S}\cdot\hat{\mathbf{n}} = \frac{1}{2}\hbar$ prior to entering the first Stern-Gerlach device. We will then assign a probability amplitude for an atom to pass along either of the $S_z = \pm\frac{1}{2}\hbar$ beams, written:

$\langle S_z = \pm\frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle =$ Probability amplitude of observing the atom to have $S_z = \pm\frac{1}{2}\hbar$ given that originally it had an $\hat{\mathbf{n}}$ component of spin $S_i = \frac{1}{2}\hbar$. (7.41)

$$=\langle\pm|S\rangle$$

where we have simplified the notation a little: $\langle S_z = \pm\frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle \to \langle\pm|S\rangle$. The atoms are then recombined and finally, after passing through the last Stern-Gerlach device, will emerge with $S_x = \pm\frac{1}{2}\hbar$. We will then write

$$\langle S_x = \pm\tfrac{1}{2}\hbar | S_z = \tfrac{1}{2}\hbar\rangle = \text{Probability amplitude of observing the atom to have } S_x = \pm\tfrac{1}{2}\hbar \text{ given that it had a } z \text{ component of spin } S_z = \tfrac{1}{2}\hbar. \tag{7.42}$$

$$= \langle S'|+\rangle$$

with a similar definition of $\langle S_x = \pm\frac{1}{2}\hbar | S_z = -\frac{1}{2}\hbar\rangle \to \langle S'|-\rangle$ where S' can have the values of $\pm\frac{1}{2}\hbar$. We can then construct the probability amplitude of measuring the x component of spin to have the value S' given that it initially the $\hat{\mathbf{n}}$ component of spin $\mathbf{S}\cdot\hat{\mathbf{n}} = S = \frac{1}{2}\hbar$ either by analogy with what applied in the two slit experiment, or by use of the more general argument of Section 7.3. Either way, this probability amplitude is given by

$$\langle S'|S\rangle = \langle S'|+\rangle\langle +|S\rangle + \langle S'|-\rangle\langle -|S\rangle. \tag{7.43}$$

Proceeding as in the two slit result, we can argue that this is a reuslt that holds for all final states $|S'\rangle$, so that we might as well 'cancel' the common factor '$\langle S'|$' in Eq. (7.43) to give a new expression for the state $|S\rangle$, that is

$$|S\rangle = |+\rangle\langle +|S\rangle + |-\rangle\langle -|S\rangle. \tag{7.44}$$

with the understanding that we can reintroduce '$\langle S'|$' for any chosen final state, yielding an expression for the probability amplitudes as needed. What Eq. (7.44) effectively represents is a 'template' into which we insert the appropriate information in order to recover the required probability amplitudes.

We have once again shown how the state of a physical system can, in a sense, be expressed in terms of other possible states of the system, with a weighting that determines the probability of observing the system in each of these other states. To see how this interpretation can be arrived at, we first of all note that since $\langle\pm|S\rangle$ is the probability amplitude of the z component of spin having the values $\pm\frac{1}{2}\hbar$ given that it initially the $\hat{\mathbf{n}}$ component of spin was $S = \frac{1}{2}\hbar$, and similarly for $\langle S'|S\rangle$, $\langle S'|\pm\rangle$, $\langle\pm|S\rangle$, then

$$\begin{aligned}
|\langle S'|S\rangle|^2 &= \text{probability of the atomic spin being in state } |S'\rangle \text{ given that it was in state } |S\rangle\\
|\langle S'|\pm\rangle|^2 &= \text{probability of the atomic spin being in the state } |S'\rangle \text{ given that it was in state } |\pm\rangle\\
|\langle\pm|S\rangle|^2 &= \text{probability of the atomic spin being in the state } |\pm\rangle \text{ given that it was in state } |S\rangle,
\end{aligned} \tag{7.45}$$

so that the coefficients $\langle +|S\rangle$ and $\langle -|S\rangle$ in Eq. (7.44) in some sense determine 'how much' of the state $|+\rangle$ is to be found in the initial state $|S\rangle$ and 'how much' of state $|-\rangle$ is to be found in $|S\rangle$. Put another way, the expression Eq. (7.44) indicates in a symbolic way the fact that when the atomic spin is prepared in its initial state $|S\rangle$, there is built into this state the *potential* of the spin to be found in the other states $|\pm\rangle$, with the chances of the particle being found in these other states being given by the coefficients $\langle\pm|S\rangle$. Once again, we see the possibility of a system in a certain state behaving as if it is in one or another of a number of other states, and to do so in a way that is probabilistic in nature.

Ex 7.1 Experiment shows that if an atom is prepared in a state where $S = \mathbf{S}\cdot\hat{\mathbf{i}} = S_x = \frac{1}{2}\hbar$ passes through a Stern-Gerlach device with a magnetic field in the z direction, it has an equal probability of exiting with either $S_z = \frac{1}{2}\hbar$ or $S_z = -\frac{1}{2}\hbar$. In other words we have, with $|S\rangle = |S_x = \frac{1}{2}\hbar\rangle$:

$$|\langle\pm|S\rangle|^2 = \tfrac{1}{2}$$

so that

$$\langle\pm|S\rangle = \frac{1}{\sqrt{2}}e^{i\phi_\pm}$$

where $\phi_\pm$ are unknown phase factors. Thus, we have

$$|S\rangle = \frac{e^{i\phi_+}}{\sqrt{2}}|+\rangle + \frac{e^{i\phi_-}}{\sqrt{2}}|-\rangle.$$

We need more information to work out what the phase factors are. We cannot do this here, but what is found is that

$$|S\rangle = e^{-i\phi}\frac{1}{\sqrt{2}}\big[|+\rangle + |-\rangle\big].$$

The last remaining phase factor remains indeterminate. We can choose any value for it without affecting the results of any calculations of physically measurable quantities, something that we cannot show just yet. We shall see this indeterminate phase factor emerge on many occasions.

This result, i.e. that the state of a quantum system can be expressed in the manner of Eq. (7.44) has come about because of the ‘sum of probability amplitudes for unobserved paths’ rule illustrated here using the spin half system as an example. The same mathematical structure appearing in Eq. (7.44) is assumed to arise in the description of all physical systems, i.e. it represents a ‘law of nature’ which ultimately leads to considering the quantum state of a system as an abstract vector in some kind of vector space, a mathematical language that allows for the possibility, realized physically, that a system in a given state can behave as if it is in some sense made up of other distinct states.