# Chapter 7

# Probability Amplitudes

EVIDENCE was presented in the preceding Chapter that under certain circumstances physical systems of quite disparate nature, but usually on the atomic scale, have the common properties of randomness and interference effects, that have no classical explanation. These general properties are the indicators, or the signatures, of the existence of basic physical laws that are not part of our normal everyday view of the way the world behaves, which immediately poses the problem of determining, and clearly stating, what these fundamental laws are. But there is in fact a second problem that has to be dealt with here. In order to state these basic laws, the question arises as to whether or not the concepts and mathematical language of classical physics is adequate for the task. We have seen that the mathematics of waves works fine for particles moving through space: from the wave function, which can be derived by solving the Schrödinger equation, information on the position and momentum, angular momentum, energy, and much else besides can be obtained. But we have also seen that the wave function cannot be used to describe spin. Apart from anything else, spin is a discrete variable, with a finite number of values, unlike the position of a particle which can vary in a continuous way. So what is needed is a mathematical language in terms of which these laws can be expressed, but which can assume outwardly different forms when applied to different physical systems. The two problems are not independent: the process of determining what the physical laws are guides us in the development of the mathematical language needed to express these laws. The outcome of all this, in another example of what has been termed 'the unreasonable effectiveness of mathematics in physics' is a mathematical language that was already well known to mathematicians in the early days of the quantum theory: the mathematics of linear vector spaces. It was a British theoretical physicist, Paul Dirac, who played a major role in formulating quantum mechanics in this way. He was inspired to do so, in part, because of the above-mentioned need for a theory that could cope with both particle spin and particle motion through space. What this work does, amongst other things, is bring into sharp focus the problem of defining what is meant by the state of a quantum system, and how a quantum state is to be represented mathematically. A further consequence of his work is a succinct notation which encapsulates both the physics and the mathematics, the Dirac bra-ket notation which is used, amongst other things, to represent the state of a quantum system. The first issue to be dealt with here is then that of coming to terms with what we mean by the state of a system, following which we look again at the two-slit experiment, which sets the scene for the development of the Dirac/Feynman version of quantum mechanics.

## 7.1 The State of a System

The notion of the state of a system is a central one in both classical and quantum physics, though it is often possible to live with only an intuitive idea of what it means. However, it proves to be important here to have the concept of the state of a system clearly defined. Ideally, specifying the state of a system would amount to gathering together all the information that it is possible

to know about the system at any instant in time. The information should be enough so that, in principle, it would be possible to reconstruct the system at some other time and place and have the reconstructed system behave in exactly the same way as the original. From the point of view of physics, this information would take the form of numerical data that specifies, for instance for a single particle, its position and momentum at any instant in time. How this information is gained is, of course, another matter, though in classical physics the accepted point of view is that the information is there to be found; it is up to us to be clever enough to get at it. Likewise, preparing a system in a particular state, that is, preparing it such that the various quantities characterising its state are given desired values, such as preparing a particle in a state with a certain position and momentum, is assumed to be possible, though not necessarily easy. Of course, the simple matter of measuring the value of, say, the position of a particle is tantamount to preparing it to have that particular value, provided the particle is not destroyed in the process of having its position measured!

Of course, many systems when considered in all their detail are forbiddingly complex, but fortunately it is not always necessary to specify *everything* about a system. For instance, if we are interested in describing the orbit of the Earth around the sun, it would be sufficient to specify the state only in so far as the position and momentum of the centre of mass of the Earth. The fact that the Earth rotates, or that grass is green is not information that is required to deal with the orbital dynamics of the Earth. Knowing what is relevant and what is not is important in setting up good models of complex systems, but given that this has been decided upon for a given system, a good definition of the state of the system consistent with our intuitive notions is then to say that the state of a system is defined by specifying the maximum amount of data that can, in principle, be known simultaneously without mutual interference or contradiction about the system.

According to classical physics, it is possible in principle, if not always in practice, to determine *exactly* all the quantities needed to specify the state of a physical system. Thus, for a system consisting of a single particle, the state of the system is specified by giving the position and the momentum of the particle at the instant of interest. For a multi-particle system, the state could be specified by giving the positions and momenta of the individual particles, or else, if there are constraints of some kind, e.g. if the particles are organized into a single rigid body then it is probably sufficient to give the position and momentum of the centre of mass, the orientation of the body, and a few other quantities such as the linear and angular momentum. In practice, of course, there will always be uncertainty in our knowledge of these quantities, but this can be put down to inadequacies in our experimental technique, measuring apparatus, or the sheer size and complexity of the system under consideration.

When it comes to actually representing the state of a classical system, there is no way of doing this that has any significance beyond simply being a list of the values of all the physical parameters that can be determined for the system, though different ways of presenting this information offer different advantages in different circumstances. An example of one way of representing the information is, for a single particle, to plot the position and momentum of the particle as a point $(x, p)$ in what is known as *phase space*. As a function of time, this point will trace out a path in phase space, this path then representing the evolution of the state of the system as a function of time.

The situation is somewhat different when quantum effects become important. It is not possible, even in principle, to specify the position *and* the momentum of a particle with total precision. The uncertainty principle tells us that we either have to compromise on the accuracy with which we specify each of the quantities, or else have to deal with the consequences of knowing, say, the position exactly, implying that the momentum is totally unknown, or vice versa. So what then are we to do with the notion of the state of a quantum system? The situation is one of accepting the information that we have about the system, and taking that as specifying the state. Doing so is consistent with the definition of state presented above, and is a reflection of what

knowledge will really do possess about the system. This is information we have without mutual interference – i.e. we specify the position of a particle, but not its momentum, or vice versa. Thus we cannot represent the state of the system in the same way that we can for a classical system, as a point in phase space, for instance. Similarly we can only specify one component of the spin of a particle, say the $x$ component of the spin of a particle, in which case we cannot specify its $y$ or $z$ component. The information should also be without contradiction, i.e. we do not claim a particle to be at one position *and* at some other. Of course, there remains the question of how we obtain this information about a quantum system, though, as in the classical case, measuring the value of some observable property of a quantum system is also tantamount to preparing it in a state in which it definitely has the measured value, provided, once again, that the system is not destroyed by the measuring process.

Here we will introduce a notation for the state of a quantum system which, for the present, is nothing more than a fancy way of writing down all the information that we can know about a quantum system, but which turns out to be very useful in describing the mathematical properties of quantum systems. The notation looks like this:

$$\left| \begin{array}{l} \text{All the data concerning the system that can be known with-} \\ \text{out mutual interference or contradiction.} \end{array} \right\rangle \tag{7.1}$$

The symbol $|\ \rangle$ is known as a *ket*. Contained within the ket is a summary of the data specifying the state of a system, and hence a ket is also referred to as the state of the system. Thus, for instance, if we know the position of a particle is $x = 3$ cm with respect to some origin, then the state would be $|x = 3\text{ cm}\rangle$. If we know that a particle has a $z$ component of spin equal to $\frac{1}{2}\hbar$, then the state would be $|S_z = \frac{1}{2}\hbar\rangle$. There would then be no such state as $|S_x = \frac{1}{2}\hbar, S_z = \frac{1}{2}\hbar\rangle$ since it is not possible to known both $S_x$ and $S_z$ simultaneously – the measurement of one component interferes with the value of any other previously measured component. We also cannot have a state like $|S_x = \frac{1}{2}\hbar, S_x = -\frac{1}{2}\hbar\rangle$, or $|x = 3cm, x = 7cm\rangle$, either of these being a contradiction.

At this stage there seems to be no point to enclosing the description of the state of a system within the symbol $|\ \rangle$, but, as will be seen later, quantum systems have the schizophrenic property of behaving as if they are *simultaneously* in a number of different states, and the way that this property of quantum systems is represented mathematically is by treating the states of a quantum system as if they are vectors in a way that we will be discussing later. Hence the above symbol is also known as a *ket vector* or *state vector*.

At times, when we want to be less specific, we would write $|x\rangle$ as being the state in which the $x$ position of a particle is known, or $|p_x\rangle$ to be the state in which the $x$ component of momentum is known, though the value in each case is unspecified. Finally, if we want to talk about a state without trying to spell out just what it is that is known precisely, i.e. an arbitrary state, we will write $|\psi\rangle$ or $|\phi\rangle$ or some such symbol. As a companion to this notation we will introduce a second way of writing down what this state is:

$$\left\langle \begin{array}{l} \text{All the data concerning the system that can be known with-} \\ \text{out mutual interference or contradiction.} \end{array} \right|. \tag{7.2}$$

This symbol is known as a *bra* or *bra vector* and is equally well a way of representing the state of a quantum system. The distinction between a bra and a ket lies in the mathematics of quantum mechanics, which we will be dealing with later.

To see how this notation is motivated and employed, and how it acquires a mathematical meaning, we return to the two slit experiment.

## 7.2 The Two Slit Experiment Revisited

We now want to recast the two slit experiment in a fashion that leads to a new way of formulating quantum mechanics. The argument to be presented here is not meant to be rigorous, but more

to suggest a way of thinking about quantum mechanics that can be made more precise, and very general in that it can be applied to any physical system. The experimental set up will be as before. Particles, which have all passed through exactly the same preparation procedure and hence are presumably all in the same state, are produced at a source $S$. These particles are then incident on a screen in which there are two narrow slits, 1 and 2, and beyond this screen is a further screen which the particles will ultimately strike, the point at which they hit this screen being registered in some way.

Since the particles are all prepared in the same way, they will presumably all be associated with the same wave function. The detailed form of the wave function will depend on many things, not the least of which is that the particles are all produced at the source $S$. So, to remind us of this, we will introduce the notation for the wave function

$$\Psi_S(x) = \text{probability amplitude of finding the particle at } x \text{ given that it originated at the source } S. \tag{7.3}$$

We will assume this wave function is unity at the source, i.e.

$$\Psi_S(S) = 1. \tag{7.4}$$

It is not important that we do this. We could choose otherwise, only to find that it cancels out at the end. Furthermore, we will not be concerning ourselves with the possible time dependence of the wave function here – in effect we are assuming some kind of steady state. This wave function will propagate through space and at the positions of the two slits will have values $\Psi_S(n)$, $n$ =1, 2. These waves will then pass through these slits on their way towards the observation screen. The task then is to determine the amplitude of the waves at a point $x$ on the observation screen due to waves that originated at each of the slits. To do that we will first of all suppose that, if the amplitude of the wave incident on slit 1 is unity, then the resulting wave amplitude at a position $x$ on the screen is $\Psi_1(x)$. But since the amplitude of the wave at slit 1 is $\Psi_S(1)$ then it appears reasonable to suppose that we scale up the amplitude of the wave from slit 1 incident on the observation screen at $x$ by the same factor, i.e. $\Psi_S(1)$, so that the amplitude of the wave incident on the screen at $x$ will be $\Psi_S(1)\Psi_1(x)$. Of course, this is an assumption, but it is what is observed for, say, light waves passing through a slit – if the amplitude of a light wave incident on a slit is doubled, then the amplitude of the wave that emerges from the slit is also doubled. Light waves are *linear*. Probability amplitudes are hence also assumed to be linear.

In a similar way, the amplitude at $x$ due to waves from slit 2 will be $\Psi_S(2)\Psi_2(x)$. Consequently, the total amplitude of the wave at $x$ will be

$$\Psi_S(x) = \Psi_S(1)\Psi_1(x) + \Psi_S(2)\Psi_2(x) \tag{7.5}$$

where $\Psi_S(x)$ is the amplitude of waves at $x$ that originated from the source $S$. This then is the probability amplitude of observing a particle at $x$ given that it originated from the source $S$, i.e. by the Born interpretation Eq. (4.15), $|\Psi_S(x)|^2 dx$ is the probability density of observing a particle in the region $x$ to $x + dx$. This result is the same as the one presented in Eq. (4.14), but here we have expressed this total probability amplitude as a sum of two contributions of the form

$$\begin{aligned} \Psi_S(n)\Psi_n(x) = &\text{ Probability amplitude of observing the particle at slit } n \\ &\text{ given that it originated from the source } S. \\ \times &\text{ Probability amplitude of observing the particle at } x \text{ given} \\ &\text{ that it originated from slit } n. \\ = &\text{ Probability amplitude for the particle to pass from the} \\ &\text{ source } S \text{ to point } x \text{ through slit } n. \end{aligned} \tag{7.6}$$

### 7.2.1 Sum of Amplitudes in Bra(c)ket Notation

It is at this point that we make use of the new notation we introduced in the preceding Section. Consider, for example, $\Psi_S(n)$, which we stated above as being the probability amplitude of observing the particle at slit $n$ given that it originated from the source $S$. We could equivalently say that $\Psi_S(n)$ is the probability amplitude of observing the particle at the position of slit $n$, given that it was originally at the position of the source $S$, or even more precisely, that it is the probability amplitude of observing the particle to be in the state in which it is at the position of slit $n$, given that it was in a state in which it was at the position of the source $S$. If we use our notation from the previous Section to write

$$\begin{aligned} |S\rangle &\equiv \text{state of the particle when at the position of the source } S \\ |n\rangle &\equiv \text{state of the particle when at the position of slit } n \\ |x\rangle &\equiv \text{state of the particle when at the position } x \end{aligned} \tag{7.7}$$

we can then write, for instance,

$$\begin{aligned} \Psi_S(n) = \ & \text{Probability amplitude of observing the particle in state } |n\rangle \\ & \text{given that it was in state } |S\rangle. \end{aligned} \tag{7.8}$$

This we will, finally, write as

$$\Psi_S(n) = \langle n||S\rangle = \langle n|S\rangle \tag{7.9}$$

i.e. we have written the final state as a bra, and where we have replaced the double vertical bar by a single bar. We can make similar replacements for the other probability amplitudes:

$$\Psi_S(x) \to \langle x|S\rangle; \quad \Psi_n(x) \to \langle x|n\rangle \tag{7.10}$$

i.e., for instance, $\langle x|S\rangle$ is the probability amplitude of finding the particle at $x$, i.e. in the state $|x\rangle$, given that it was initially at the source $S$, i.e. in the state $|S\rangle$. Recalling that a symbol such as $|S\rangle$ is also known as a ket, we can trace the origin of the names bra and ket to the fact that $\langle x|S\rangle$ can be thought of as a quantity enclosed in a pair of angled brackets, or 'bra(c)kets'. In terms of this new notation, the above result Eq. (7.5) becomes

$$\langle x|S\rangle = \langle x|1\rangle\langle 1|S\rangle + \langle x|2\rangle\langle 2|S\rangle. \tag{7.11}$$

Being able to write the probability amplitude in this way is a particularly important result as it leads directly to a new way of looking at the idea of the state of a physical system that lies at the heart of quantum mechanics.

### 7.2.2 Superposition of States for Two Slit Experiment

We can note that the expression Eq. (7.11) will hold true for all values of the variable $x$, i.e. it does not hold true for just one value of $x$. Because of this, we can do something rather radical and that is 'cancel' the $\langle x|$ to leave

$$|S\rangle = |1\rangle\langle 1|S\rangle + |2\rangle\langle 2|S\rangle \tag{7.12}$$

with the understanding that what we have created by doing so is a template into which we insert $\langle x|$ when we so desire, and thereby regain the expression Eq. (7.11). As a result of this step, we have apparently given a new meaning to the ket $|S\rangle$ as being more than just a way of summarizing all the information on the state of the system. The expression just obtained seems to suggest that there is a deeper mathematical and perhaps physical meaning that can be assigned to $|S\rangle$. For instance, we are free to manipulate Eq. (7.12) as we see fit. For instance we could solve for $|2\rangle$ to give

$$|2\rangle = \frac{|S\rangle - |1\rangle\langle 1|S\rangle}{\langle 2|S\rangle} \tag{7.13}$$

(recall that the $\langle \ldots | \ldots \rangle$ are all just complex numbers) and so on, and then put $\langle x|$ back in to give

$$\langle x|2\rangle = \frac{\langle x|S\rangle - \langle x|1\rangle\langle 1|S\rangle}{\langle 2|S\rangle}. \tag{7.14}$$

However, this development in the notation has more to offer than just this. Having created this new kind of expression, it is worthwhile to see whether or not it can be given any useful meaning. The new interpretation is a very potent one, and constitutes the central feature of quantum mechanics, the idea that a system, in some sense, can be simultaneously in a number of different physical states, otherwise known as a 'superposition of states'.

To see how this interpretation can be arrived at, we first of all note that since $\langle n|S\rangle$ is the probability amplitude of finding the particle at slit $n$, given that it was initially at the source $S$, and similarly for $\langle x|S\rangle$, $\langle x|n\rangle$, $\langle n|S\rangle$, then

$$\begin{aligned} |\langle x|S\rangle|^2 &= \text{probability of finding the particle in state } |x\rangle \text{ given that it was in state } |S\rangle \\ |\langle x|n\rangle|^2 &= \text{probability of finding the particle in state } |x\rangle \text{ given that it was in state } |n\rangle \\ |\langle n|S\rangle|^2 &= \text{probability of finding the particle in state } |n\rangle \text{ given that it was in state } |S\rangle, \end{aligned} \tag{7.15}$$

so that the coefficients $\langle 1|S\rangle$ and $\langle 2|S\rangle$ in Eq. (7.12) in some sense determine 'how much' of the state $|1\rangle$ is to be found in the initial state $|S\rangle$ and 'how much' of state $|2\rangle$ is to be found in $|S\rangle$. Put another way, the expression Eq. (7.12) indicates in a symbolic way the fact that when the particle is prepared in its initial state $|S\rangle$, there is built into this state the *potential* of the particle being found in the other states $|n\rangle$, with the chances of the particle being found in these other states being given by the coefficients $\langle n|S\rangle$. It is this sort of result that lies at the heart of quantum mechanics – the idea that a system in a certain state can behave as if it is in one or another of a number of other states, and to do so in a way that is probabilistic in nature. It is also out of this expression that the basic mathematical formalism of quantum mechanics is built, the idea being that the kets $|n\rangle$ can be considered to be, in some sense, vectors, much like the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ encountered in mechanics, and the coefficients $\langle n|S\rangle$ are the components of the total vector $|S\rangle$ along these two directions. It is this analogue that we will pursue in some detail in later Chapters. But for the present, we will show that the result Eq. (7.11) which we have 'derived' in the case of the two slit experiment can also be shown (and much more rigorously) to be an intrinsic part of the Stern-Gerlach experiment. In fact Eq. (7.11) is of a general form that is taken, ultimately, as a central postulate of quantum mechanics, one that is suggested by circumstantial evidence, though not proven. In effect, it is a law of nature.

## 7.3 The Stern-Gerlach Experiment Revisited

The aim here is to show that a result equivalent to the sum of amplitudes for different paths result obtained for the two slit experiment can also be shown to arise in the Stern-Gerlach experiment. However, unlike the previous result for the two slit arrangement, the result to be obtained here can be shown to be more rigorously based and more useful in the long run as providing the insight necessary to generalize the result to any physical system.

### 7.3.1 Probability amplitudes for particle spin

Further development then follows what was done in Section 7.2 in that we use the sum of probability amplitudes result to arrive at the notion of the spin state of the system being expressible as a 'superposition' of other possible states.

Without affecting the generality of the argument, we will specialize to the case illustrated in Fig. (7.1).

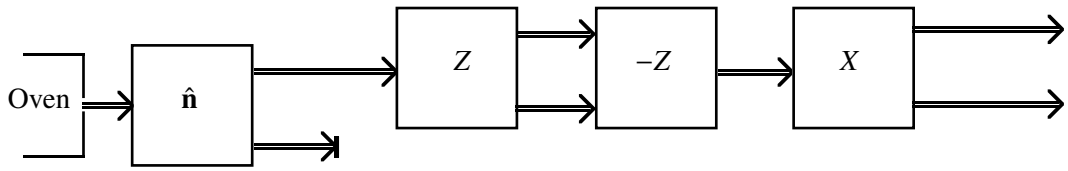


Figure 7.1: Atomic beam for which $S_i = \frac{1}{2}\hbar$ split into $S_z = \pm\frac{1}{2}\hbar$ beams and then recombined before passing through final Stern-Gerlach device with magnetic field in $x$ direction.

This is the same setup as presented in Fig. (6.9) except that here the atom can emerge from the first Stern-Gerlach device in one or the other of two separate beams corresponding to the atomic spin component $S_i = \mathbf{S}\cdot\hat{\mathbf{n}} = \pm\frac{1}{2}\hbar$ where $\hat{\mathbf{n}}$ is some arbitrary orientation of the magnetic field (in the $XZ$ plane) in the device. The atoms in one of the beams ($S_i = \frac{1}{2}\hbar$) is then selected and passed through a Stern-Gerlach device where the magnetic field further separates this beam according to its $z$ component of spin. The beams are then recombined before entering the final Stern-Gerlach apparatus, this arrangement being made so as to erase the information on which beam the each atom emerged from the preceding apparatus.

We are now going to make use of the analogy of this set-up with the two slit interference experiment, and the assumption that the observed measurement outcomes at the final Stern-Gerlach apparatus are of quantum origin, to construct assign a probability interpretation to the observed behaviour of particle spin. Thus, we will assume that the oven plus the first Stern-Gerlach device is the equivalent of the source of identically prepared particles in the two slit experiment, except that here the atoms are all identically prepared to have $S_i = \mathbf{S}\cdot\hat{\mathbf{n}} = \frac{1}{2}\hbar$. The next Stern-Gerlach device is analogous to the two slits in that the atoms can, in principle, emerge along two different paths corresponding to $S_z = \pm\frac{1}{2}\hbar$. We can tell which path an atom follows (i.e. via the $S_z = \frac{1}{2}\hbar$ or the $S_z = -\frac{1}{2}\hbar$ beam) by monitoring which beam an atom emerges from after it passes through the first $z$ oriented Stern-Gerlach device in much the same way that we can monitor which slit a particle passes through in the two slit experiment.

We will then assign a probability amplitude for an atom to pass along either of the $S_z = \pm\frac{1}{2}\hbar$ beams, written:

$$\begin{aligned}\langle S_z = \pm\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle = &\quad \text{Probability amplitude of observing the atom to have } S_z = \\ &\quad \pm\tfrac{1}{2}\hbar \text{ given that originally it had an } \hat{\mathbf{n}} \text{ component of spin} \\ &\quad S_i = \tfrac{1}{2}\hbar. \\ = &\langle \pm | S \rangle\end{aligned} \tag{7.16}$$

where we have simplified the notation a little: $\langle S_z = \pm\frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle \rightarrow \langle \pm | S\rangle$.

The atomic beams are then recombined and finally, after passing through the last Stern-Gerlach device, emerge with $S_x = \pm\frac{1}{2}\hbar$. We then write

$$\begin{aligned}\langle S_x = \pm\tfrac{1}{2}\hbar | S_z = \tfrac{1}{2}\hbar\rangle = &\quad \text{Probability amplitude of observing the atom to have } S_x = \\ &\quad \pm\tfrac{1}{2}\hbar \text{ given that it had a } z \text{ component of spin } S_z = \tfrac{1}{2}\hbar. \\ = &\langle S' | + \rangle\end{aligned} \tag{7.17}$$

with a similar definition of $\langle S_x = \pm\frac{1}{2}\hbar | S_z = -\frac{1}{2}\hbar\rangle \rightarrow \langle S' | - \rangle$ where $S'$ can have the values of $\pm\frac{1}{2}\hbar$. We can then construct the probability amplitude of measuring the $x$ component of spin to have the

value $S'$ given that it initially the $\hat{\mathbf{n}}$ component of spin $\mathbf{S}\cdot\hat{\mathbf{n}} = S = \frac{1}{2}\hbar$ either by analogy with what applied in the two slit experiment, or by use of the more general argument of Sections 7.3.3 and 8.3.2. Either way, this probability amplitude is given by

$$\langle S'|S\rangle = \langle S'|+\rangle\langle +|S\rangle + \langle S'|-\rangle\langle -|S\rangle. \tag{7.18}$$

We now want to obtain form this result the idea of the spin state of the atoms emerging from the first Stern-Gerlach apparatus as being 'made up of' the states $|\pm\rangle$.

### 7.3.2 Superposition of States for Spin Half

Proceeding as in the two slit result, we can argue that this is a result that holds for all final states $|S'\rangle$, so that we might as well 'cancel' the common factor '$\langle S'|$' in Eq. (7.18) to give a new expression for the state $|S\rangle$, that is

$$|S\rangle = |+\rangle\langle +|S\rangle + |-\rangle\langle -|S\rangle. \tag{7.19}$$

with the understanding that we can reintroduce '$\langle S'|$' for any chosen final state, yielding an expression for the probability amplitudes as needed. What Eq. (7.19) effectively represents is a 'template' into which we insert the appropriate information in order to recover the required probability amplitudes.

We have once again shown how the state of a physical system can, in a sense, be expressed in terms of other possible states of the system, with a weighting that determines the probability of observing the system in each of these other states. To see how this interpretation can be arrived at, we first of all note that since $\langle\pm|S\rangle$ is the probability amplitude of the $z$ component of spin having the values $\pm\frac{1}{2}\hbar$ given that it initially the $\hat{\mathbf{n}}$ component of spin was $S = \frac{1}{2}\hbar$, and similarly for $\langle S'|S\rangle$, $\langle S'|\pm\rangle$, $\langle\pm|S\rangle$, then

$$\begin{aligned}
|\langle S'|S\rangle|^2 &= \text{probability of the atomic spin being in state } |S'\rangle \text{ given that} \\
&\quad \text{it was in state } |S\rangle \\
|\langle S'|\pm\rangle|^2 &= \text{probability of the atomic spin being in the state } |S'\rangle \text{ given} \\
&\quad \text{that it was in state } |\pm\rangle \\
|\langle \pm|S\rangle|^2 &= \text{probability of the atomic spin being in the state } |\pm\rangle \text{ given} \\
&\quad \text{that it was in state } |S\rangle,
\end{aligned} \tag{7.20}$$

so that the coefficients $\langle +|S\rangle$ and $\langle -|S\rangle$ in Eq. (7.19) in some sense determine 'how much' of the state $|+\rangle$ is to be found in the initial state $|S\rangle$ and 'how much' of state $|-\rangle$ is to be found in $|S\rangle$. Put another way, the expression Eq. (7.19) indicates in a symbolic way the fact that when the atomic spin is prepared in its initial state $|S\rangle$, there is built into this state the *potential* of the spin to be found in the other states $|\pm\rangle$, with the chances of the particle being found in these other states being given by the coefficients $\langle\pm|S\rangle$. Once again, we see the possibility of a system in a certain state behaving as if it is in one or another of a number of other states, and to do so in a way that is probabilistic in nature.

### 7.3.3 A derivation of sum-over-probability-amplitudes for spin half

The task here is to show that the sum of probability amplitudes expression Eq. (7.18) can be extracted in the case of spin half from the assumed experimentally observed spin probability. This is a purely mathematical exercise which, nevertheless, has a clear physical interpretation in terms of a particular kind of Stern-Gerlach experimental arrangement that is analogous of the two slit experiment. As a bonus, it turns out that it is also possible to derive explicit expressions for spin half probability amplitudes.

Here will consider a fairly general kind of experimental arrangement illustrated in Fig. (7.2) in which an atomic beam is passed through two consecutive Stern-Gerlach devices. It should be noted here that in contrast to, for instance, Fig. (6.9), a slightly different choice of axes has been made. The choice made is such that the angle of orientation of the magnetic fields is the $\phi$ angular coordinate of a set of spherical polar coordinates. Thus, it will be assumed that each Stern-Gerlach device will have their magnetic fields lying in the $XY$ plane, so that the beam of atoms is heading in the $z$ direction. Apart from that, the magnetic fields of the two devices can be oriented in an arbitrary direction, $\hat{\mathbf{n}}$ for the first, and $\hat{\mathbf{m}}$ for the second.

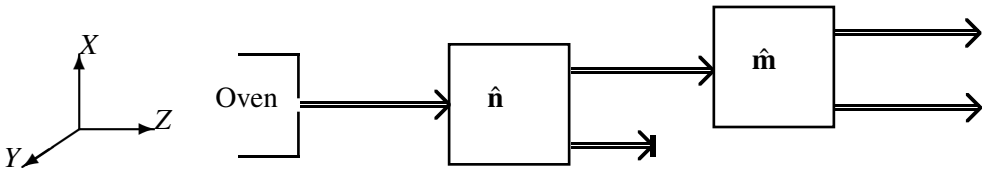


Figure 7.2: Similar to Fig. (6.4) but with magnetic field of second Stern-Gerlach device allowed to be oriented in an arbitrary direction $\hat{\mathbf{m}}$ direction.

Here, the first magnetic field makes an angle of $\phi_i$ with the $x$ direction and the second an angle of $\phi_f$ with the $x$ direction, see Fig. (7.3).

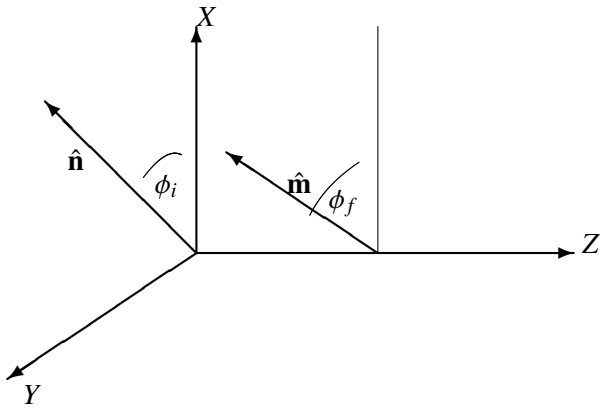


Figure 7.3: Orientation of the magnetic fields in the two Stern-Gerlach devices in Fig. (7.2)

The angle between the directions of the two magnetic fields is $\phi = \phi_i - \phi_f$. Consequently, the probability of an atom emerging in the beam for which $S_f = \mathbf{S}\cdot\hat{\mathbf{m}} = \frac{1}{2}\hbar$, given that it entered the last Stern-Gerlach device with $S_i = \mathbf{S}\cdot\hat{\mathbf{n}} = \frac{1}{2}\hbar$ will be, from Eq. (6.29) (using $\cos\phi = \cos(-\phi)$)

$$P(S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar) = \cos^2(\tfrac{1}{2}\phi) = \cos^2[(\phi_f - \phi_i)/2]. \tag{7.21}$$

The question we want to address here is the following: is it possible to show that this probability (which we are assuming has been determined by experiment) is in any way to be understood as being of purely quantum mechanical origin?

One way of doing this is to show that this probability is consistent with its interpretation as the modulus squared of a possibly complex probability amplitude, and moreover, that this complex probability amplitude can be written as the sum over probability amplitudes associated with different ways that the spin of the atom can arrive at its final observed value, in a way analogous to the two slit experiment. Thus, we begin this investigation by writing

$$P(S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar) = |\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle|^2 \tag{7.22}$$

where $\langle S_f = \frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$ is, following the notation used in the previous Section, the probability amplitude of observing the $\hat{\mathbf{m}}$ component of the spin of an atom to be $\frac{1}{2}\hbar$ given that the $\hat{\mathbf{n}}$ component is known to be $\frac{1}{2}\hbar$.

We now want to show that $\langle S_f = \frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$ can indeed be written as a sum of contributions associated with probability amplitudes of the spins 'passing through' some other states, as in Eq. (7.11) for the two slit case, prior to the final measurement of $S_f$. These 'other states' would be those in which the atom has a spin component $S_I = \mathbf{S}\cdot\hat{\mathbf{l}} = \pm\frac{1}{2}\hbar$ in some direction $\hat{\mathbf{l}}$. So, by analogy with the two slit result, what we are aiming to do here is to show that

$$\begin{aligned}\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle =& \langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle \\ &+ \langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle\end{aligned} \tag{7.23}$$

which is to be interpreted as meaning that the probability amplitude $\langle S_f = \frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$ of an atom being found in state $|S_f = \frac{1}{2}\hbar\rangle$ given that it was initially in the state $|S_i = \frac{1}{2}\hbar\rangle$ is the sum of the probability amplitudes of the atomic spin 'passing through', without observation, the two intermediate states $|S_I = \pm\frac{1}{2}\hbar\rangle$.

This is, of course, a purely mathematical exercise, but rather remarkably it nevertheless leads to a result that can be given meaning in terms of the experimental set-up shown in Fig. (7.4)). This is a set-up which involves using a Stern-Gerlach device with magnetic field in the direction $\hat{\mathbf{l}}$ aligned at some angle $\phi_I$. Atoms exiting such a device will emerge with spin components $S_I = \mathbf{S}\cdot\hat{\mathbf{l}}$ that, as usual, can have the two values $\pm\frac{1}{2}\hbar$. In a sense, these two beams are much like the slits in the two slit experiment. A second device in the direction $-\hat{\mathbf{l}}$ merely serves to bring the two beams back together again prior to entering the final Stern-Gerlach device.

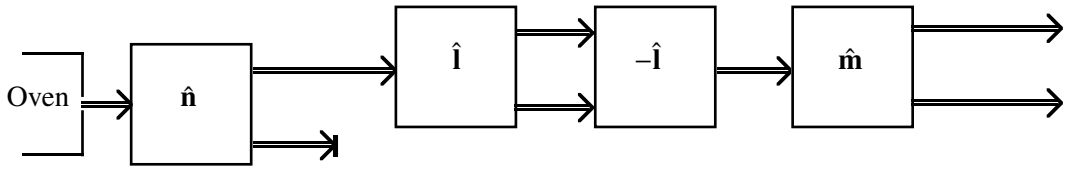


Figure 7.4: Atoms exiting from the first Stern-Gerlach apparatus with $S_i = \frac{1}{2}\hbar$ and entering the second can emerge in either of two ($S_l = \pm\frac{1}{2}\hbar$) beams. The beams are then recombined, so erasing information on which beam any individual atom might emerge in, before passing through a final Stern-Gerlach device which measures the component of the atomic spin in the $\hat{\mathbf{m}}$ direction.

What this setup does is to present the experimenter with the opportunity of measuring any component of atomic spin (in the $XY$ plane only, in this case) of an atom, prior to a final measurement of $S_f$. As we shall see, if no observation is made as to which beam each atom emerges from the second Stern-Gerlach device, then the final observed results for $S_f$ are the same as if the intervening Stern-Gerlach apparatus was not present at all, while if an intervening observation *is* made, then the observed result is consistent with the absence of interference effects.

That the result Eq. (7.23) is implied by the probability Eq. (7.21) requires us to write down the possible mathematical form for the probability amplitude $\langle S_f = \frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$. Starting with Eq. (7.21) and Eq. (7.22)

$$P(S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar) = \cos^2[(\phi_f - \phi_i)/2] = |\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle|^2 \tag{7.24}$$

it follows that

$$\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle = \cos[(\phi_f - \phi_i)/2]e^{i\varphi(\phi_f - \phi_i)} \tag{7.25}$$

where $\exp(i\varphi(\phi_f - \phi_i))$ is a phase factor which is unknown except that $\varphi$ is a function of the angular separation $\phi_f - \phi_i$ only — it is only the difference in the angular settings of the Stern-Gerlach devices that can be of physical significance. From this we have

$$\cos[(\phi_f - \phi_i)/2] = \langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar \rangle e^{-i\varphi(\phi_f - \phi_i)} \tag{7.26}$$

a result that we will make use of shortly. But before that, we need to write this cos factor in a way that will lead to an interpretation of the spin system passing through an intermediate Stern-Gerlach apparatus with its magnetic field oriented at an angle $\phi_I$. We can do this by expanding out the cos term by writing

$$\cos[\tfrac{1}{2}(\phi_f - \phi_i)] = \cos[\tfrac{1}{2}(\phi_f - \phi_I) + \tfrac{1}{2}(\phi_I - \phi_i)] \tag{7.27}$$

and using standard trigonometric formulae to give

$$\begin{aligned}&\cos[\tfrac{1}{2}(\phi_f - \phi_I) + \tfrac{1}{2}(\phi_I - \phi_i)] \\ &\qquad = \cos[\tfrac{1}{2}(\phi_f - \phi_I)]\cos[\tfrac{1}{2}(\phi_I - \phi_i)] - \sin[\tfrac{1}{2}(\phi_f - \phi_I)]\sin[\tfrac{1}{2}(\phi_I - \phi_i)].\end{aligned} \tag{7.28}$$

The second, sin dependent term becomes, on further use of simple trigonometry:

$$\sin[\tfrac{1}{2}(\phi_f - \phi_I)]\sin[\tfrac{1}{2}(\phi_I - \phi_i)] = -\cos[\tfrac{1}{2}(\phi_f - (\phi_I + \pi))]\cos[\tfrac{1}{2}((\phi_I + \pi) - \phi_i)] \tag{7.29}$$

so that overall we get

$$\begin{aligned}&\cos[(\phi_f - \phi_i)/2] \\ &\qquad = \cos[\tfrac{1}{2}(\phi_f - \phi_I)]\cos[\tfrac{1}{2}(\phi_I - \phi_i)] + \cos[\tfrac{1}{2}(\phi_f - (\phi_I + \pi))]\cos[\tfrac{1}{2}((\phi_I + \pi) - \phi_i)].\end{aligned} \tag{7.30}$$

If we now make use of Eq. (7.26) (with corresponding expressions for $\langle S_I = \frac{1}{2}\hbar | S_i = \pm\frac{1}{2}\hbar\rangle$) and so on, we can write this as

$$\begin{aligned}&\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar \rangle e^{-i\varphi(\phi_f - \phi_i)} \\ &= \langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar \rangle e^{-i\varphi(\phi_f - \phi_I)} \langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar \rangle e^{-i\varphi(\phi_I - \phi_i)} \\ &+ \langle S_f = \tfrac{1}{2}\hbar | S_i = -\tfrac{1}{2}\hbar \rangle e^{-i\varphi(\phi_f - (\phi_I + \pi))} \langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar \rangle e^{-i\varphi((\phi_I + \pi) - \phi_i)}.\end{aligned} \tag{7.31}$$

To get the required result Eq. (7.23), the various phase factors must cancel, which requires

$$e^{-i\varphi(\phi_f - \phi_i)} = e^{-i\varphi(\phi_f - \phi_I)} e^{-i\varphi(\phi_I - \phi_i)} = e^{-i\varphi(\phi_f - (\phi_I + \pi))} e^{-i\varphi((\phi_I + \pi) - \phi_i)}. \tag{7.32}$$

If we, for instance, put $\alpha = \phi_f - \phi_I$ and $\beta = \phi_i - \phi_I$, and put $F(x) = \exp(-i\varphi(x))$, then the first of these equations can be written

$$F(\alpha - \beta) = F(\alpha)F(\beta) \tag{7.33}$$

and hence $F(x)$ must be an exponential function whose exponent is linear in $x$:

$$F(x) = e^{-iax} = e^{-i\varphi(x)} \tag{7.34}$$

where $a$ is a constant. Hence

$$\varphi(x) = ax + 2n\pi \tag{7.35}$$

where $n$ is an integer. The term $2n\pi$ is the same in all the phase factors, so it will always cancel out, so we might as well set $n$ to zero. Thus we end up with

$$\cos[(\phi_f - \phi_i)/2] = \langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar \rangle e^{-ia(\phi_f - \phi_i)} \tag{7.36}$$

or

$$\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar \rangle = \cos[(\phi_f - \phi_i)/2] e^{ia(\phi_f - \phi_i)} \tag{7.37}$$

and similarly for the other cos factors.

We can pin down the phase factor a little further by noting that in $\langle S_f = -\frac{1}{2}\hbar | S_i = \frac{1}{2}\hbar\rangle$ we could replace $\phi_f \to \phi_f + 2\pi$ without changing the physical meaning of the result. Thus we must have

$$\cos[(\phi_f - \phi_i)/2]e^{ia(\phi_f-\phi_i)} = \cos[(\phi_f - \phi_i + 2\pi)/2]e^{ia(\phi_f-\phi_i+2\pi)} \tag{7.38}$$

and hence that

$$e^{2\pi a i} = -1. \tag{7.39}$$

which can be satisfied if we put $a = n + \frac{1}{2}$ where $n$ is an integer that can be chosen freely and is conventionally set equal to $-1$ so that

$$\langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle = e^{-i(\phi_f-\phi_i)/2}\cos[\tfrac{1}{2}(\phi_f - \phi_i)] \tag{7.40}$$

So overall we have simply shown that Eq. (7.30) can be written

$$\begin{aligned} e^{-i(\phi_f-\phi_i)/2}\cos[\tfrac{1}{2}(\phi_f - \phi_i)] &= e^{-i(\phi_f-\phi_I)/2}\cos[\tfrac{1}{2}(\phi_f - \phi_I)]e^{-i(\phi_I-\phi_i)/2}\cos[\tfrac{1}{2}(\phi_I - \phi_i)] \\ &+ e^{-i(\phi_f-\phi_I-\pi)/2}\cos[\tfrac{1}{2}(\phi_f - (\phi_I + \pi))]e^{-i(\phi_I+\pi-\phi_i)/2}\cos[\tfrac{1}{2}((\phi_I + \pi) - \phi_i)] \end{aligned} \tag{7.41}$$

which becomes, using Eq. (7.40)

$$\begin{aligned} \langle S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle =& \langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle \\ &+ \langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle \end{aligned} \tag{7.42}$$

as required. We also have from this the probability of measuring an atom to be in a state for which $S_f = \frac{1}{2}\hbar$:

$$\begin{aligned} &P(S_f = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar) \\ &= \left|\langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle + \langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle\right|^2 \\ &= \left|\langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle\right|^2 + \left|\langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle\right|^2 \\ &\quad + 2\mathrm{Re}\left[\langle S_f = \tfrac{1}{2}\hbar | S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle(\langle S_f = \tfrac{1}{2}\hbar | S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar | S_i = \tfrac{1}{2}\hbar\rangle)^*\right] \end{aligned} \tag{7.43}$$

where the third term is an *interference* term. Thus we have managed to show that the probability distribution associated with passing spin half systems through a Stern-Gerlach device to prepare the spin of the atoms to have a value $S_i = \frac{1}{2}\hbar$, and then through a second to measure the spin components $S_f$ can be directly interpreted as being the consequence of the interference of probability amplitudes in a way analogous to the two slit experiment.

An important feature of the result Eqs. (7.23) and (7.42) is that it is expressed in terms of the probability amplitudes of the system passing through one or the other of the intermediate states for which $S_I = \mathbf{S}\cdot\hat{\mathbf{I}} = \pm\frac{1}{2}\hbar$, i.e. there appears here probability amplitudes that would be associated with the use of an intermediate Stern-Gerlach device to separate the atoms according to their $S_I$ component of spin. But this result was in a sense 'built-in' to the original mathematical expression for the probability Eq. (7.21), i.e. no mention was made of any actual intermediate Stern-Gerlach device when deriving the result Eq. (7.42). In other words, the observation probability, Eq. (7.43) can be written in a way that looks like there are interference contributions associated with the possibility of passing the atoms through an intervening Stern-Gerlach device, *even though no such Stern-Gerlach apparatus need ever appear in the actual experiment.* In fact, if we *do* insert the appropriate device with magnetic field in the $\hat{\mathbf{I}}$ direction to separate and then recombine the two beams, we find that *if there is no information available that specifies through which of the*

$S_I = \pm\frac{1}{2}\hbar$ *beams the atoms pass*, then the probabilities for observing $S_f = \pm\frac{1}{2}\hbar$ for the recombined beam are exactly the same state as if the extra device were not present, i.e. the probability is given by $\cos^2(\frac{1}{2}(\phi_f - \phi_i))$, Eq. (7.21), which is independent of the intermediate states that appear in Eq. (7.42). Put another way, having the intermediate device present, but not observing what it does, is the same as not having it there at all. On the other hand if we *do* have information as to which beam the atoms come through, then the observation probability is

$$P(S_f = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar) = |\langle S_f = \tfrac{1}{2}\hbar|S_I = \tfrac{1}{2}\hbar\rangle\langle S_I = \tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar\rangle|^2 + |\langle S_f = \tfrac{1}{2}\hbar|S_I = -\tfrac{1}{2}\hbar\rangle\langle S_I = -\tfrac{1}{2}\hbar|S_i = \tfrac{1}{2}\hbar\rangle|^2 \quad (7.44)$$

where we have lost the interference terms of Eq. (7.43). This result *does* depend on the intermediate state that happens to be observed. These considerations show that built into the properties of the state of the spin system is the *potential* for the system to be observed in *any* other state $|S_I = \frac{1}{2}\hbar\rangle$, for which the relevant probability amplitude $\langle S_I = \frac{1}{2}\hbar|S_i = \frac{1}{2}\hbar\rangle$ is non-zero. It is this property of the states of quantum systems that leads to the formulation of the state of a quantum system as a vector belonging to a complex vector space: the language of vectors makes it possible to express one vector (i.e. state) in terms of other vectors (i.e. states).

## 7.4 The General Case of Many Intermediate States

The above analysis has only been presented for two cases, the two slit experiment and spin half, but both these examples can be readily generalized. Thus, for instance, we could consider an interference experiment in which there are multiple slits in the first barrier. The final result for the probability amplitude of observing an electron striking the screen at position $x$ after having set out from a source $S$ will then be given by

$$\langle x|S\rangle = \sum_{n=1}^{N}\langle x|n\rangle\langle n|S\rangle \quad (7.45)$$

where we have supposed that there are $N$ slits in the first barrier, so that the electron can pass through $N$ intermediate states. Likewise, if we repeat the Stern-Gerlach experiment with spin 1 atoms, we would find that the atoms would emerge in one or the other of three beams corresponding to $S_z = -\hbar, 0, \hbar$. More generally, if the atoms have spin $s$ (where $s = 0$ or $\frac{1}{2}$ or 1 or $\frac{3}{2}, \ldots$) then they will emerge in one of a total of $2s + 1$ different possible beams corresponding to $S_z = -s\hbar$, or $(-s + 1)\hbar$, or $\ldots$, or $(s - 1)\hbar$, or $s\hbar$. We can then expect that we could write

$$\langle S'|S\rangle = \sum_{n=1}^{2s+1}\langle S'|n - s - 1\rangle\langle n - s - 1|S\rangle \quad (7.46)$$

i.e. the atom can pass through $2s + 1$ intermediate states, where we have written $|n\rangle$ for the state in which the atom has a $z$ component of spin $S_z = n\hbar$. In either case, the 'cancellation' trick could then be applied to give, for instance in the latter case,

$$|S\rangle = \sum_{n=-s}^{s}|n\rangle\langle n|S\rangle \quad (7.47)$$

so we recover a generalization of the result found for spin half, Eq. (7.19) that we ultimately will interpret as a relationship between vectors.

We now want to move beyond these two particular cases. In effect, we want to extract from the above results concepts that can be applied to *any* physical system in the hope that what is obtained is a fundamental quantum principle that would apply to all physical systems, i.e. it would constitute a new law of nature.

It is clearly the case in the above examples that it is not the probability of occurrence of random events that is fundamental, rather it is probability amplitudes. The probabilities are derived from probability amplitudes – they are obtained by squaring an appropriately determined probability amplitude. Furthermore, these probability amplitudes satisfy a particular law concerning the role of intermediate states: the probability amplitude $\langle\phi|\psi\rangle$ for finding a system in a state $|\phi\rangle$ given that it was in the state $|\psi\rangle$ can be written in a way analogous to the particular examples considered here, that is, as a sum over the probability amplitudes associated with all the different 'pathways' by which a system can make its way from the initial state $|\psi\rangle$ to the final state $|\phi\rangle$ via a collection of intermediate states, i.e. we can always write

$$\langle\phi|\psi\rangle = \sum_n \langle\phi|n\rangle\langle n|\psi\rangle \tag{7.48}$$

It is this relationship that constitutes the fundamental law of quantum mechanics. Its natural consequence, obtained by taking the sum over intermediate states concept to its logical extreme, is the path integral formulation of quantum mechanics due to Feynman, the 'version' of quantum mechanics that can be applied to essentially any physical system whatsoever, including the universe itself, though it can yield mathematical formulae to which it can be very difficult to give a meaning. But it also yields the familiar forms of quantum mechanics such as wave mechanics and the Schrödinger equation.

Our aims here are much more modest, so we turn to the important consequence of Eq. (7.48) that we have already made much use of, that is the result obtained by the cancellation trick:

$$|\phi\rangle = \sum_n |n\rangle\langle n|\psi\rangle. \tag{7.49}$$

This is an expression that can be interpreted as describing the physical fact that any system in a given state can behave as if it is in some sense made up of other distinct states, here, the 'intermediate states' $|n\rangle$ with the probability amplitudes providing a weighting to be given to the different possibilities in the sense that, $|\langle n|\psi\rangle|^2$ is the probability that the system could be observed to be in the state $|n\rangle$. The challenge is to identify the 'intermediate states' for any given system. Moreover, this relationship can be given an interpretation in which the state of a quantum system can be considered as an abstract vector in some kind of vector space, i.e. we can set up a mathematical description of the quantum properties of a physical system in terms of the mathematical formalism of complex vector spaces.

## 7.5 Probabilities vs probability amplitudes

The fundamental formula of quantum mechanics, Eq. (7.48) has a very close resemblance to a well known formula of classical probability theory. We can analyse the 'sum-over-paths' picture according to these rules. So suppose, as we have above we have a system initially prepared in some initial state $|\psi\rangle$, and we are interested in calculating the probability of finding the system in some final state $|\phi\rangle$. (If we had never heard of quantum mechanics, we would have to consider the Dirac notation manner of writing down the state of a system as nothing much more than a rather fancy way of writing out some data). Further suppose that the system can access this final state by different pathways via some intermediate states $|n\rangle$. Then, according to classical probability theory, the probability of finding the system in state $|\phi\rangle$ given that it was originally in the state $|\psi\rangle$ is given by

$$P_{cl}(\phi|\psi) = \sum_n P(\phi|n)P(n|\psi)$$

i.e. the system starts in the state $|\psi\rangle$, and has the probability $P(n|\psi)$ that it will end up in the state $|n\rangle$, and once it 'arrives there' we then have the probability $P(\phi|n)$ that it will then end up in the

state $|\phi\rangle$ given that it was in the state $|n\rangle$. Multiplying these two probabilities will then give the probability of the system starting from $|\psi\rangle$ and passing through $|n\rangle$ on its way to the final state $|\phi\rangle$. We then sum over all the possible intermediate states $|n\rangle$ to give the total probability of arriving in the state $|\phi\rangle$. However, what is found experimentally is that *if the system is never observed in any of the intermediate states* $|n\rangle$, this probability is not given by this classical result – the measurements show evidence of interference effects, which can only be understood if this probability is given as the square of a 'probability amplitude'.

Thus, we find we must write $P(\phi|\psi) = |\langle\phi|\psi\rangle|^2$, where $\langle\phi|\psi\rangle$ is a complex number, generally referred to as the probability amplitude of finding the system in state $|\phi\rangle$ given that it was in state $|\psi\rangle$, and this probability amplitude is then given by

$$\langle\phi|\psi\rangle = \sum_n \langle\phi|n\rangle\langle n|\psi\rangle \tag{7.50}$$

where the sum is over the probability amplitudes of the system passing through all the possible intermediate states $|n\rangle$.

If we square this result we find that

$$\begin{aligned} P(\phi|\psi) = |\langle\phi|\psi\rangle|^2 &= \sum_n |\langle\phi|n\rangle|^2|\langle n|\psi\rangle|^2 + \text{cross terms} \\ &= \sum_n P(\phi|n)P(n|\psi) + \text{cross terms}. \end{aligned} \tag{7.51}$$

We note that this expression consists, in part, of the classical result Eq. (7.5) but there is, in addition, cross terms. It is these terms that are responsible for the interference effects such as those observed in the two slit experiment, or in the Stern-Gerlach experiment. If we observe which intermediate state the system 'passes through' it is always found that these interference terms are washed out, reducing the result to the classical result Eqn. (7.5). This we have seen in the case of two slit interference wherein if the slit through which the particle passes is observed, then the interference pattern on the observation screen is washed out.

We can see how the general comments made above look in the case of a spin half system for which we have

$$\langle S'|S\rangle = \langle S'|+\rangle\langle +|S\rangle + \langle S'|-\rangle\langle -|S\rangle. \tag{7.52}$$

To then calculate the probability of measuring the atomic spin to have the value $S' = \frac{1}{2}\hbar$, given that it was initially known to have the value $S = \frac{1}{2}\hbar$ requires us to calculate the square of the probability amplitude $\langle S'|S\rangle$:

$$\begin{aligned} P(S'|S) &= \left|\langle S'|S\rangle\right|^2 \\ &= \left|\langle S'|+\rangle\langle +|S\rangle\right|^2 + \left|\langle S'|-\rangle\langle -|S\rangle\right|^2 + 2\text{Re}\left[\langle S'|+\rangle\langle +|S\rangle\langle S|-\rangle\langle -|S'\rangle\right] \end{aligned} \tag{7.53}$$

where the last term is the interference term. But if we observe, or measure, the $z$ component of the atomic spin *before* we measure its final spin $S'$, then we have seen above that the interference term is wiped out, so that we get

$$P_{\text{cl}}(S'|S) = \left|\langle S'|+\rangle\langle +|S\rangle\right|^2 + \left|\langle S'|-\rangle\langle -|S\rangle\right|^2 = \left|\langle S'|+\rangle\right|^2|\langle +|S\rangle|^2 + \left|\langle S'|-\rangle\right|^2|\langle -|S\rangle|^2 \tag{7.54}$$

$$= P(S'|+)P(+|S) + P(S'|-)P(-|S). \tag{7.55}$$

It is this result that has an immediate interpretation according to the rules of classical probability theory:

| | | | |
|---|---|---|---|
| Probability of measuring the spin to be $S' = \frac{1}{2}\hbar$ given that it was initially $S' = \frac{1}{2}\hbar$ | = | Probability of measuring the spin to be $S' = \frac{1}{2}\hbar$ given it was observed to be $S_z = \frac{1}{2}\hbar$ | × Probability of measuring the spin to be $S_z = \frac{1}{2}\hbar$ given it was initially $S' = \frac{1}{2}\hbar$ |
| | + | Probability of measuring the spin to be $S' = \frac{1}{2}\hbar$ given it was observed to be $S_z = -\frac{1}{2}\hbar$ | × Probability of measuring the spin to be $S_z = -\frac{1}{2}\hbar$ given it was initially $S = \frac{1}{2}\hbar$ |

i.e. this is the way we would have calculated the probability if we assumed the familiar rules of classical physics applied. Instead, quantum physics tells us that we must use what is almost the same rule, except that the rule is applied to probability amplitudes, not probabilities, i.e. Eq. (7.5) is replaced by

| | | | |
|---|---|---|---|
| Probability amplitude of measuring the spin to be $S' = \frac{1}{2}\hbar$ given that it was initially $S' = \frac{1}{2}\hbar$ | = | Probability amplitude of measuring the spin to be $S' = \frac{1}{2}\hbar$ given it was observed to be $S_z = \frac{1}{2}\hbar$ | × Probability amplitude of measuring the spin to be $S_z = \frac{1}{2}\hbar$ given it was initially $S' = \frac{1}{2}\hbar$ |
| | + | Probability amplitude of measuring the spin to be $S' = \frac{1}{2}\hbar$ given it was observed to be $S_z = -\frac{1}{2}\hbar$ | × Probability amplitude of measuring the spin to be $S_z = -\frac{1}{2}\hbar$ given it was initially $S = \frac{1}{2}\hbar$ |

which is just Eq. (7.52) written out in words. Thus, in a sense, what quantum mechanics provides us with is a different set of rules for calculating the probability of outcome of random events. But it is more than that. The classical rule refers to outcomes that actually occur, the quantum rule refers to possibilities, or potentials for possible outcomes which are never realized in actuality. But these possibilities nevertheless exert a 'ghostly influence', in that they give rise to interference effects which distinguish the fully quantum result from the classically expected result.