PHYSICS 301 QUANTUM PHYSICS I (2007)

Assignment 3 Solutions

1. (a) With respect to a pair of orthonormal vectors $|1\rangle$ and $|2\rangle$ that span the state space $\mathcal{H}$ of a certain system, the Hermitean operator $\widehat{Q}$ is defined by its action on these base states as follows:

$$\widehat{Q}|1\rangle = 2|1\rangle - 2i|2\rangle \qquad \widehat{Q}|2\rangle = 2i|1\rangle - |2\rangle.$$

(i) What is the matrix represention of $\widehat{Q}$ in the $\{|1\rangle, |2\rangle\}$ basis?

(ii) Show that the states

$$|q_1\rangle = \frac{1}{\sqrt{5}}\big(|1\rangle + 2i|2\rangle\big) \qquad |q_2\rangle = \frac{1}{\sqrt{5}}\big(2|1\rangle - i|2\rangle\big).$$

are eigenstates of $\widehat{Q}$ and that the associated eigenvalues are $q_1 = -2$ and $q_2 = 3$ respectively.

(b) The operator $\widehat{Q}$ above represents a certain physical observable Q of a quantum system which is prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|q_1\rangle + \frac{1+i}{\sqrt{3}}|q_2\rangle.$$

(i) What are the possible results of a measurement of the observable Q?

(ii) What are the probabilities of obtaining each of the possible results?

(iii) What is the state of the system *after* the measurement is performed for each of the possible measurement outcomes?

SOLUTION

(a) (i) The matrix representation is given by

$$\widehat{Q} \doteq \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

where $Q_{ij} = \langle i|\widehat{Q}|j\rangle$.
By direct calculation from the above defining properties of $\widehat{Q}$, it follows that

$$\widehat{Q} \doteq \begin{pmatrix} 2 & 2i \\ -2i & -1 \end{pmatrix}$$

(ii) The calculation can proceed by either using the matrix representation of the operator, or by carrying out the calculation in bra-ket notation. The first will be done in the bra-ket notation. Thus, what is required is

$$\begin{aligned}\widehat{Q}|q_1\rangle =& \widehat{Q}\frac{1}{\sqrt{5}}\left(|1\rangle + 2i|2\rangle\right)\\ =&\frac{1}{\sqrt{5}}\left(\widehat{Q}|1\rangle + 2i\widehat{Q}|2\rangle\right)\\ =&\frac{1}{\sqrt{5}}\left[2|1\rangle - 2i|2\rangle + 2i(2i|1\rangle - |2\rangle)\right]\\ =&\frac{1}{\sqrt{5}}\Big[-2|1\rangle - 4i|2\rangle\Big]\\ =&-2\frac{1}{\sqrt{5}}\Big[|1\rangle + 2i|2\rangle\Big]\\ =&-2|q_1\rangle.\end{aligned}$$

Thus, $|q_1\rangle$ is an eigenstate of $\widehat{Q}$ with eigenvalue -2.
The second case will be dealt with using the column vector representation of $|q_2\rangle$, i.e.

$$|q_2\rangle \doteq \begin{pmatrix}\langle 1|q_2\rangle\\ \langle 2|q_2\rangle\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\ -i\end{pmatrix}.$$

Thus

$$\widehat{Q}|q_2\rangle = \begin{pmatrix}2 & 2i\\ -2i & -1\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}2\\ -i\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}6\\ -3i\end{pmatrix} = 3\frac{1}{\sqrt{5}}\begin{pmatrix}2\\ -i\end{pmatrix} \doteq 3|q_2\rangle$$

i.e. $|q_2\rangle$ is an eigenstate of $\widehat{Q}$ with eigenvalue 3.

(b) (i) The possible results of a measurement of the observable Q are the eigenvalues of $\widehat{Q}$, i.e. -2 and 3.

(ii) The probability amplitude of obtaining the result $q_1 = -2$ is

$$\langle q_1|\psi\rangle = \frac{1}{\sqrt{3}}\langle q_1|q_1\rangle + 0 = \frac{1}{\sqrt{3}}\langle q_1|q_1\rangle$$

where the orthogonality of the eigenvectors with different eigenvalues has been used to eliminate the second term. Further

$$\langle q_1|q_1\rangle = \frac{1}{5}\begin{pmatrix}1 & -2i\end{pmatrix}\begin{pmatrix}1\\ 2i\end{pmatrix} = 1$$

so that $|q_1\rangle$ is normalized to unity. Thus

$$\langle q_1|\psi\rangle = \frac{1}{\sqrt{3}}$$

and hence the probability of obtaining the result $q_1 = 2$ is

$$|\langle q_1|\psi\rangle|^2 = \frac{1}{3}.$$

The probability amplitude of obtaining the result $q_2 = 3$ is

$$\langle q_2|\psi\rangle = \frac{1+i}{\sqrt{3}}\langle q_2|q_2\rangle$$

where once again the orthogonality of the eigenstates has been used. Checking that $|q_2\rangle$ is normalized to unity:

$$\langle q_2|q_2\rangle = \frac{1}{5}\begin{pmatrix}2 & i\end{pmatrix}\begin{pmatrix}2\\ -i\end{pmatrix} = 1$$

then yields for the probability of obtaining the result $q_2 = 3$:

$$|\langle q_2|\psi\rangle|^2 = \left|\frac{1+i}{\sqrt{3}}\right|^2 = \frac{2}{3}.$$

(iii) If the result obtained is $q_1 = -2$, then the system ends up in the state $|q_1\rangle$, and if the result $q_2 = 3$ is obtained, then the system ends up in the state $|q_2\rangle$.

---

2. In so-called isospin theory, the neutron and the proton are assumed to be two different states, $|n\rangle$ and $|p\rangle$ respectively, of the one particle called the nucleon. Suppose as a result of a collision betwen a nucleon and another particle, the state of the nucleon undergoes a change represented by the operator $\widehat{E}$ defined by

$$\widehat{E}|n\rangle = \big(|n\rangle + |p\rangle/\sqrt{2}$$
$$\widehat{E}|p\rangle = \big(|n\rangle - |p\rangle\big)/\sqrt{2}$$

'mixes' the two states $|n\rangle$ and $|p\rangle$.

(a) Suppose the nucleon is prepared in the state $|n\rangle$, and the nucleon suffers such a collision. What is the probability that the nucleon could still be observed to be a neutron after the collision?

(b) Construct a matrix to represent $\widehat{E}$ in the $\{|n\rangle, |p\rangle\}$ basis.

(c) Write the state $|\psi\rangle = \big(|n\rangle - 2i|p\rangle\big)/\sqrt{5}$ as a column vector, and determine the new state of the system after the collision has occurred.

(d) Write the bra vector $\langle\psi|$ as a row vector, and hence show that the probability the nucleon could be observed in the state $|\psi\rangle$ after the collision is non-zero. Evaluate this probability.

(e) Do there exist states of the nucleon for which the collision does not change the state? If so, determine what the state or states are that have this property.

SOLUTION

(a) If prepared in the state $|n\rangle$, then the state after the collision is

$$\widehat{E}|n\rangle = \big(|n\rangle + |p\rangle\big)/\sqrt{2}.$$

It is readily seen that this state is normalized to unity. Hence the probability amplitude that the nucleon will still be observed to be a neutron after the collision will be

$$\langle n|\widehat{E}|n\rangle = \langle n|\big(|n\rangle + |p\rangle\big)/\sqrt{2} = \frac{1}{\sqrt{2}}$$

and hence the probability will be

$$|\langle n|\widehat{E}|n\rangle|^2 = \tfrac{1}{2}.$$

(b) This matrix will be

$$\widehat{E} \doteq \begin{pmatrix} E_{nn} & E_{np} \\ E_{pn} & E_{pp} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

where, for instance

$$\begin{aligned} E_{nn} =& \langle n|\widehat{E}|n\rangle \\ =& \langle n|\big(|n\rangle + |p\rangle\big)/\sqrt{2} = \frac{1}{\sqrt{2}} \end{aligned}$$

and so on.

(c) As a column vector, the state vector $|\psi\rangle$ is

$$|\psi\rangle \doteq \begin{pmatrix} \langle n|\psi\rangle \\ \langle p|\psi\rangle \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{5}} \\ -\dfrac{2i}{\sqrt{5}} \end{pmatrix}.$$

After the collision has occurred, the new state is

$$\widehat{E}|\psi\rangle \doteq \begin{pmatrix} \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \dfrac{1}{\sqrt{5}} \\ -\dfrac{2i}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1-2i \\ 1+2i \end{pmatrix}$$

This state can also readily shown to be normalized to unity.

(d) The bra vector, as a row vector, is written

$$\langle\psi| = \begin{pmatrix} \dfrac{1}{\sqrt{5}} & \dfrac{2i}{\sqrt{5}} \end{pmatrix}$$

and the probability amplitude of the nucleon being found in the state $|\psi\rangle$ after the collision will be $\langle\psi|\widehat{E}|\psi\rangle$, and given by

$$\langle\psi|\widehat{E}|\psi\rangle = \begin{pmatrix} \dfrac{1}{\sqrt{5}} & \dfrac{2i}{\sqrt{5}} \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 1-2i \\ 1+2i \end{pmatrix} = \frac{1}{\sqrt{50}}\big(1-2i+2i-4\big) = -\frac{3}{\sqrt{50}}$$

and hence the probability of the nucleon being found in the state $|\psi\rangle$ after the collision will be

$$|\langle\psi|\widehat{E}|\psi\rangle|^2 = \frac{9}{50} = 0.18$$

(e) For a state of the system to be unchanged by the action of $\widehat{E}$ this would require the state to be mapped into a multiple of itself – recall that a given state vector, or any multiple of that state vector will represent the same physical state of the system. But this requirement is identical to requiring the state to be an eigenstate of $\widehat{E}$. Thus we have to determine what state or states, call them $|\lambda\rangle$, are such that

$$\widehat{E}|\lambda\rangle = \lambda|\lambda\rangle.$$

The eigenvalues of $\widehat{E}$ are found in the usual way, i.e. by solving

$$\widehat{E}|\lambda\rangle = \lambda|\lambda\rangle.$$

The characteristic equation

$$\begin{vmatrix} \dfrac{1}{\sqrt{2}} - \lambda & \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0$$

has the solutions $\lambda = \pm 1$ which we will write as $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenstate with eigenvalue $\lambda_1 = 1$ then satisfies

$$\begin{pmatrix} \frac{1}{\sqrt{2}} - 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - 1 \end{pmatrix} \begin{pmatrix} \langle n|\lambda_1\rangle \\ \langle p|\lambda_1\rangle \end{pmatrix} = 0$$

which yields the two equations

$$\begin{aligned} \left(\frac{1}{\sqrt{2}} - 1\right)\langle n|\lambda_1\rangle + \frac{1}{\sqrt{2}}\langle p|\lambda_1\rangle =& 0 \\ \frac{1}{\sqrt{2}}\langle n|\lambda_1\rangle - \left(\frac{1}{\sqrt{2}} + 1\right)\langle p|\lambda_1\rangle =& 0. \end{aligned}$$

These equations are not linearly independent – at best we can solve only for the ratio of $\langle n|\lambda_1\rangle$ and $\langle p|\lambda_1\rangle$ or alternatively put $\langle p|\lambda_1\rangle = 1$ to give

$$\langle n|\lambda_1\rangle = 1 + \sqrt{2}$$

and hence the unnormalized eigenstate $|\lambda_1\rangle$ is given by

$$|\lambda_1\rangle \doteq \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}$$

which when normalized to unity becomes

$$|\lambda_1\rangle \doteq \frac{1}{\sqrt{2(2+\sqrt{2})}} \begin{pmatrix} 1 + \sqrt{2} \\ 1 \end{pmatrix}.$$

After a little bit of trigonometry, it is possible to show that this can be written as

$$|\lambda_1\rangle \doteq \begin{pmatrix} \cos(\pi/8) \\ \sin(\pi/8) \end{pmatrix}$$

The other eigenstate is obtained in a similar fashion. The eigenstate with eigenvalue $\lambda_2 = -1$ satisfies

$$\begin{pmatrix} \frac{1}{\sqrt{2}} + 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} + 1 \end{pmatrix} \begin{pmatrix} \langle n|\lambda_2\rangle \\ \langle p|\lambda_2\rangle \end{pmatrix} = 0$$

which yields the two equations

$$\begin{aligned} \left(\frac{1}{\sqrt{2}} + 1\right)\langle n|\lambda_2\rangle + \frac{1}{\sqrt{2}}\langle p|\lambda_2\rangle =& 0 \\ \frac{1}{\sqrt{2}}\langle n|\lambda_2\rangle - \left(\frac{1}{\sqrt{2}} - 1\right)\langle p|\lambda_2\rangle =& 0. \end{aligned}$$

Putting $\langle p|\lambda_2\rangle = 1$ gives

$$\langle n|\lambda_2\rangle = 1 - \sqrt{2}$$

and hence the unnormalized eigenstate $|\lambda_2\rangle$ is given by

$$|\lambda_2\rangle \doteq \begin{pmatrix} 1 - \sqrt{2} \\ 1 \end{pmatrix}$$

which when normalized to unity becomes

$$|\lambda_2\rangle \doteq \frac{1}{\sqrt{2(2-\sqrt{2})}}\begin{pmatrix}1-\sqrt{2}\\ \\ 1\end{pmatrix}.$$

This, too, can be written in trignometric form as

$$|\lambda_2\rangle \doteq \begin{pmatrix}-\sin(\pi/8)\\ \\ \cos(\pi/8)\end{pmatrix}$$

The above states λ_n, $n = 1, 2$ are then the required states for the nucleon that would remain unchanged through the action of $\widehat{E}$.

---

3. (a) The negative oxygen molecular ion O_2^- consists of a pair of oxygen atoms separated by a distance $2a$. As an approximation, the electron can be assumed to be found only on one or the other of the oxygen atoms, at $x = \pm a$.

   (i) Within this approximation, what are the eigenvalues of the position operator $\widehat{x}$ for the electron?

   (ii) The Hamiltonian $\widehat{H}$ for the O_2^- ion is such that:

$$\widehat{H}|+a\rangle = \tfrac{1}{2}E\Big(|+a\rangle + e^{i\pi/4}|-a\rangle\Big)$$
$$\widehat{H}|-a\rangle = \tfrac{1}{2}E\Big(\alpha|+a\rangle + |-a\rangle\Big)$$

   where E is a real number and α is a complex number. What must the value of α be and why must it have this value?

   (b) For the correct value of α, this operator can be shown to have the eigenstates

$$|E_1\rangle = \frac{1}{\sqrt{2}}\Big(|+a\rangle + e^{i\pi/4}|-a\rangle\Big) \quad \text{and} \quad |E_2\rangle = \frac{1}{\sqrt{2}}\Big(|+a\rangle - e^{i\pi/4}|-a\rangle\Big).$$

   Show, by directly calculating $\widehat{H}|E_i\rangle$, that the associated eigenvalues are $E_1 = E$ and $E_2 = 0$.

   (c) The quantum system is prepared in the state $|\psi\rangle = \frac{1}{\sqrt{3}}\big[|+a\rangle + i\sqrt{2}|-a\rangle\big]$. A measurement is made of the energy and the result E is obtained.

   (i) What is the state of the system *after* this measurement was performed?

   (ii) If, after the above result for the energy was obtained, the position of the electron was measured, what is the probability of obtaining the result $+a$?

   (iii) What would be the probability of getting the result $-a$ if the position of the electron was measured when the system was in the original state $|\psi\rangle$?

   (iv) Could the system ever be found in a state in which the position of the electron was $x = +a$ and the energy was E? Explain your answer.

SOLUTION

(a) (i) The eigenvalues of the position operator $\widehat{x}$ will be $\pm a$.

(ii) The Hamiltonian is an observable of the system, and as such must be a Hermitean operator. Consequently, it must be the case that $\langle +a|\widehat{H}|-a\rangle = \langle -a|\widehat{H}|+a\rangle^*$. From the expressions for the action of $\widehat{H}$ on the states $|\pm a\rangle$ we have that

$$\langle +a|\widehat{H}|-a\rangle = \tfrac{1}{2}E\langle +a|\Big(\alpha|+a\rangle + |-a\rangle\Big) = \tfrac{1}{2}E\alpha$$

while

$$\langle -a|\widehat{H}|+a\rangle = \tfrac{1}{2}E\langle -a|\Big(|+a\rangle + e^{i\pi/4}|-a\rangle\Big) = \tfrac{1}{2}Ee^{i\pi/4}.$$

Consequently, the requirement that $\langle +a|\widehat{H}|-a\rangle = \langle -a|\widehat{H}|+a\rangle^*$ tells us that

$$\tfrac{1}{2}E\alpha = \big(\tfrac{1}{2}Ee^{i\pi/4}\big)^*$$

i.e.

$$\alpha = e^{-i\pi/4}.$$

(b) We require, firstly

$$\begin{aligned}
\widehat{H}|E_1\rangle =& \frac{1}{\sqrt{2}}\widehat{H}\Big(|+a\rangle + e^{i\pi/4}|-a\rangle\Big)\\
=& \frac{1}{\sqrt{2}}\Big(\widehat{H}|+a\rangle + e^{i\pi/4}\widehat{H}|-a\rangle\Big)\\
=& \frac{E}{2\sqrt{2}}\Big(|+a\rangle + e^{i\pi/4}|-a\rangle + e^{i\pi/4}\big(e^{-i\pi/4}|+a\rangle + |-a\rangle\big)\Big)\\
=& \frac{E}{\sqrt{2}}\Big(|+a\rangle + e^{i\pi/4}|-a\rangle\Big)\\
=& E|E_1\rangle,
\end{aligned}$$

i.e. $E_1 = E$. Secondly

$$\begin{aligned}
\widehat{H}|E_2\rangle =& \frac{1}{\sqrt{2}}\widehat{H}\Big(|+a\rangle - e^{i\pi/4}|-a\rangle\Big)\\
=& \frac{1}{\sqrt{2}}\Big(\widehat{H}|+a\rangle - e^{i\pi/4}\widehat{H}|-a\rangle\Big)\\
=& \frac{E}{2\sqrt{2}}\Big(|+a\rangle + e^{i\pi/4}|-a\rangle - e^{i\pi/4}\big(e^{-i\pi/4}|+a\rangle + |-a\rangle\big)\Big)\\
=& 0|E_2\rangle
\end{aligned}$$

i.e. $E_2 = 0$.

(c) (i) As the result E was obtained, the state of the system immediately after the measurement was performed is $|E_1\rangle$, i.e. the eigenstate of $\widehat{H}$ with eigenvalue E.

(ii) As the system is now in the state $|E_1\rangle$, the probability of obtaining the result $+a$ if a measurement of position was made is given by $|\langle +a|E_1\rangle|^2$. From the expression above for $|E_1\rangle$, this is readily seen to be

$$|\langle +a|E_1\rangle|^2 = \tfrac{1}{2}.$$

(iii) The probability of getting the result $-a$ if a measurement of the position of the electron was made when the system was in the state $|\psi\rangle$ is $|\langle -a|\psi\rangle|^2$. From the above expression for $|\psi\rangle$ this is readily seen to be

$$|\langle -a|\psi\rangle|^2 = \tfrac{1}{3}.$$

(iv) Suppose, after a measurement of energy, yielding the result E a measurement of the position of the electron was performed, and suppose that the result $+a$ was obtained, as in part (ii) above. The system is now in the state $|+a\rangle$. Now perform another measurement of energy. There are two possible results: E and 0. The first will occur with a probability $|\langle E_1| + a\rangle|^2 = \frac{1}{2}$, the second with a probability $|\langle E_2| + a\rangle|^2 = \frac{1}{2}$. In other words, there is a chance that the result observed for the energy of the system is no longer E as obtained on the first measurement – the result 0 is equally likely. Thus, the intervening measurement of the postiion of the electron has 'scrambled' the preceding result for the energy of the system – it is not possible for the position of the electron and the energy of the system to be known precisely at the same time.

---

4. The ammonia molecule consists of a plane of hydrogen atoms arranged in an equilateral triangle, with the nitrogen atom positioned symmetrically either above or below this plane, thereby forming a triangular pyramid shape. If we let $|1\rangle$ and $|2\rangle$ be the position eigenstates for the nitrogen atom, corresponding to the atom being either above or below the plane of hydrogen atoms respectively, then the Hamiltonian matrix for the molecule can be shown to be, in the $\{|1\rangle, |2\rangle\}$ basis

$$\widehat{H} \doteq \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}.$$

(a) Assuming that the state of the system at time t can be expressed as

$$|\psi(t)\rangle = C_1(t)|1\rangle + C_2(t)|2\rangle,$$

write down the Schrödinger equation for this system in matrix form.

(b) Confirm, by direct substitution into the equations for $C_1(t)$ and $C_2(t)$ that the solutions for these coefficients are

$$C_1(t) = \tfrac{1}{2}e^{-iE_0t/\hbar}\left(ae^{iAt/\hbar} + be^{-iAt/\hbar}\right)$$
$$C_2(t) = \tfrac{1}{2}e^{-iE_0t/\hbar}\left(ae^{iAt/\hbar} - be^{-iAt/\hbar}\right)$$

where a and b are unknown constants.

(c) The system is initially in the state $|1\rangle$. Solve for the probability of observing the system in state $|2\rangle$ at a later time t, and provide a physical interpretation for your result.

SOLUTION

(a) In matrix form, the Schrödinger equation

$$\widehat{H}|\psi\rangle = i\hbar\frac{d|\psi\rangle}{dt}$$

becomes

$$\begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = i\hbar\begin{pmatrix} \dot{C}_1 \\ \dot{C}_2 \end{pmatrix}$$

(b) Taking the derivative of the expressions given for $C_1(t)$ and $C_2(t)$ gives

$$\begin{aligned} i\hbar \begin{pmatrix} \dot{C}_1 \\ \dot{C}_2 \end{pmatrix} &= \tfrac{1}{2} \begin{pmatrix} a(E_0 - A)e^{-i(E_0-A)t/\hbar} + b(E_0 + A)e^{-i(E_0+A)t/\hbar} \\ a(E_0 - A)e^{-i(E_0-A)t/\hbar} - b(E_0 + A)e^{-i(E_0+A)t/\hbar} \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} E_0\left(ae^{-i(E_0-A)t/\hbar} + be^{-i(E_0+A)t/\hbar}\right) - A\left(ae^{-i(E_0-A)t/\hbar} - be^{-i(E_0+A)t/\hbar}\right) \\ E_0\left(ae^{-i(E_0-A)t/\hbar} - be^{-i(E_0+A)t/\hbar}\right) - A\left(ae^{-i(E_0-A)t/\hbar} + be^{-i(E_0+A)t/\hbar}\right) \end{pmatrix} \\ &= \tfrac{1}{2} \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix} \begin{pmatrix} ae^{-i(E_0-A)t/\hbar} + be^{-i(E_0+A)t/\hbar} \\ ae^{-i(E_0-A)t/\hbar} - be^{-i(E_0+A)t/\hbar} \end{pmatrix} \\ &= \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \end{aligned}$$

hence the given functions are indeed solutions of the Schrödinger equation.

(c) If the system was initially in the state $|1\rangle$, then the intial values of C_1 and C_2 will be

$$C_1 = 1, \quad C_2(0) = 0$$

which tells us that

$$a + b = 2, \quad a - b = 0$$

and hence

$$a = b = 1.$$

The probability of finding the system in the state $|2\rangle$ is then

$$|\langle 2|\psi(t)\rangle|^2 = |C_2(t)|^2.$$

Using the inital values of a and b we have

$$C_2(t) = \tfrac{1}{2}e^{-iE_0t/\hbar}\left(e^{iAt/\hbar} - e^{-iAt/\hbar}\right) = e^{-iE_0t/\hbar}\sin(At/\hbar)$$

so that

$$|\langle 2|\psi(t)\rangle|^2 = \sin^2(At/\hbar).$$

This probability starts off at zero, as it should as the system is in the other state $|1\rangle$ at this time. It thereafter oscillates with a frequency $2A/\hbar$, i.e. with a period $T = \pi\hbar/A$. After half a period, the probability of being in the state $|1\rangle$ decreases to zero – the system is, with certainty, in the state $|2\rangle$, and a half period later, the probability of being in the state $|1\rangle$ returns to unity, and so on and on . . . . ..
