MACQUARIE UNIVERSITY

Department of Physics
Division of ICS

PHYSICS 301 Quantum Mechanics (2006)

Assignment 3

Due Date: 12 April 2006

1. In so-called isospin theory, the neutron and the proton are assumed to be two different states, $|n\rangle$ and $|p\rangle$ respectively, of the one particle called the nucleon. Suppose as a result of a collision betwen a nucleon and another particle, the state of the nucleon undergoes a change represented by the operator $\hat{E}$ defined by

$$\hat{E}|n\rangle = \big(|n\rangle + i|p\rangle\big)/\sqrt{2}$$
$$\hat{E}|p\rangle = \big(i|n\rangle + |p\rangle\big)/\sqrt{2}$$

which 'mixes' the two states $|n\rangle$ and $|p\rangle$.

   (a) Suppose a neutron suffers such a collision. What is the probability that the nucleon could still be observed to be a neutron after the collision?

   (b) Construct the matrix representation of $\hat{E}$ in the $\{|n\rangle, |p\rangle\}$ basis.

   (c) (i) Write the state $|\psi\rangle = \big(|n\rangle - 2i|p\rangle\big)/\sqrt{5}$ as a column vector, and determine the new state of the system after the collision has occurred.

   (ii) Write the bra vector $\langle\psi|$ as a row vector, and hence show that the probability the nucleon could be observed in the state $|\psi\rangle$ after the collision is non-zero. Evaluate this probability.

   (d) Show that $\hat{E}\hat{E}^\dagger = \hat{1}$ where $\hat{1}$ is the unit operator.

   (e) If the nucleon is in an arbitrary state $|\psi\rangle = a|n\rangle + b|p\rangle$ before a collision, what is its state after the collision? If $|\psi\rangle$ is normalized to unity, show that the state of the system after the collision is still normalized to unity.

   (f) Do there exist states of the nucleon for which the collision does not change the state? If so, determine what the state or states are that have this property.

2. (a) With respect to a pair of orthonormal vectors $|1\rangle$ and $|2\rangle$ that span the state space $\mathcal{H}$ of a certain system, the operator $\hat{Q}$ is defined by its action on these base states as follows:

$$\hat{Q}|1\rangle = 2|1\rangle + 2i|2\rangle \qquad \hat{Q}|2\rangle = \alpha|1\rangle - |2\rangle.$$

   where α is a quantity to be determined.

   (i) What is the matrix represention of $\hat{Q}$ in the $\{|1\rangle, |2\rangle\}$ basis?

   (ii) $\hat{Q}$ is known to be an observable of the system. What is the value of α, and why does it have this value?

   (iii) Show that the states

$$|q_1\rangle = \frac{1}{\sqrt{5}}\big(|1\rangle - 2i|2\rangle\big) \qquad |q_2\rangle = \frac{1}{\sqrt{5}}\big(2|1\rangle + i|2\rangle\big).$$

   are eigenstates of $\hat{Q}$ and that the associated eigenvalues are $q_1 = -2$ and $q_2 = 3$ respectively.

(b) The system is prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{1+i}{\sqrt{3}}|2\rangle.$$

(i) What are the possible results of a measurement of the observable Q?

(ii) What are the probabilities of obtaining each of the possible results?

(iii) What is the state of the system *after* the measurement is performed for each of the possible measurement outcomes?

3. (a) The negative oxygen molecular ion O_2^- consists of a pair of oxygen atoms separated by a distance $2a$. As an approximation, the electron can be assumed to be found only on one or the other of the oxygen atoms, at $x = \pm a$.

   (i) Within this approximation, write down the eigenvalue equation for the position operator $\hat{x}$ for the electron.

   (ii) The momentum $\hat{p}$ for the electron can be represented by a 2×2 matrix.

   A. Why would the matrix be 2×2 in size?

   B. Given that the matrix representing the momentum is, in the position representation

$$\hat{p} \doteq p_0 \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}$$

   show that the eigenvalues are $\pm p_0$ and prove that the associated eigenvectors $|\pm p_0\rangle$ of $\hat{p}$ are given by

$$|p_0\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ ie^{i\phi} \end{pmatrix} \quad \text{and} \quad |-p_0\rangle \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -ie^{i\phi} \end{pmatrix}.$$

   C. Show that the states $|\pm p_0\rangle$ are orthonormal.

(b) The quantum system is prepared in the state $|\psi\rangle = \frac{1}{\sqrt{3}}\left[|+a\rangle + i\sqrt{2}|-a\rangle\right]$. A measurement is made of the momentum and the result p_0 is obtained.

   (i) What is the state of the system *after* this measurement was performed?

   (ii) If, after the above result for the momentum was obtained, the position of the electron was measured, what is the probability of obtaining the result $+a$?

   (iii) What would be the probability of getting the result $-a$ if the position of the electron was measured when the system was in the original state $|\psi\rangle$?

   (iv) Could the system ever be found in a state in which the position of the electron was $x = +a$ and the momentum was p_0? Explain your answer.

4. For the O_2^- of the previous question, the Hamiltonian $\hat{H}$ is such that:

$$\begin{aligned} \widehat{H}|+a\rangle &= \tfrac{1}{2}E\Big(|+a\rangle + e^{i\phi}|-a\rangle\Big) \\ \widehat{H}|-a\rangle &= \tfrac{1}{2}E\Big(e^{-i\phi}|+a\rangle + |-a\rangle\Big) \end{aligned}$$

(a) Write down the matrix representing $\hat{H}$ in the position representation.

(b) Assuming that the state of the system at time t can be expressed as

$$|\psi(t)\rangle = C_+(t)|+a\rangle + C_-(t)|-a\rangle,$$

write down the Schrödinger equation for this system in matrix form.

(c) Confirm, by direct substitution into the equations for $C_1(t)$ and $C_2(t)$ that the solutions for these coefficients are

$$C_+(t) = \tfrac{1}{2}\left(ae^{-i\omega t} + b\right)$$
$$C_-(t) = \tfrac{1}{2}e^{-i\phi}\left(ae^{-i\omega t} - b\right)$$

where a and b are unknown constants and $\omega = E/\hbar$.

(d) The system is initially, at $t = 0$ in the state $|-a\rangle$. Solve for the probability of observing the system in state $|+a\rangle$ at a later time t.

(e) At what time $t = T$ would the probability of the electron being observed on the oxygen atom at $+a$ first be a maximum?

(f) Assuming it is valid to do so, analyse this result classically to estimate the momentum that the electron would have to have in order to cross from the left hand to the right hand atom in time T. [It turns out that the momentum of the electron can have the magnitude $p_0 = mEa/\hbar$. Your result here will be slightly different.]