



Wave Mechanics Notes for PHYS201

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Chapter 1

Introduction

The world of our every-day experiences – the world of the not too big (compared to, say, a galaxy), and the not too small, (compared to something the size and mass of an atom), and where nothing moves too fast (compared to the speed of light) – is the world that is mostly directly accessible to our senses. This is the world usually more than adequately described by the theories of classical physics that dominated the nineteenth century: Newton's laws of motion, including his law of gravitation, Maxwell's equations for the electromagnetic field, and the three laws of thermodynamics. These classical theories are characterized by, amongst other things, the notion that there is a 'real' world out there, one that has an existence independent of ourselves, in which, for instance, objects have a definite position and momentum which we could measure to any degree of accuracy, limited only by our experimental ingenuity. According to this view, the universe is evolving in a way completely determined by these classical laws, so that if it were possible to measure the positions and momenta of all the constituent particles of the universe, and we knew all the forces that acted between the particles, then we could in principle predict to what ever degree of accuracy we desire, exactly how the universe (including ourselves) will evolve. Everything is predetermined – there is no such thing as free will, there is no room for chance. Anything apparently random only appears that way because of our ignorance of all the information that we would need to have to be able to make precise predictions.

This rather gloomy view of the nature of our world did not survive long into the twentieth century. It was the beginning of this century which saw the formulation of a new set of fundamental principles that provides a framework into which all physical theories must fit. To a greater or lesser extent all natural phenomena appear to be governed by these principles which together constitute what is undoubtedly the most successful theory of modern physics: quantum mechanics. One of the crucial consequences of quantum mechanics was the realization that the world view implied by classical physics, as outlined above, was no longer tenable. Irreducible randomness was built into the laws of nature. The world is inherently probabilistic in that events can happen without a cause, a fact first stumbled on by Einstein, but never fully accepted by him. But more than that, quantum mechanics admits the possibility of an interconnectedness or an 'entanglement' between physical systems, even those possibly separated by vast distances, that has no analogue in classical physics, and which plays havoc with our strongly held presumptions that there is an objectively real world 'out there' that has properties independent of us as observers.

Quantum mechanics is often thought of as being the physics of the very small, but this is true only insofar as the fact that peculiarly quantum effects are most readily observed at the atomic level. But in the everyday world that we usually experience, where the

classical laws of Newton and Maxwell seem to be able to explain so much, it quickly becomes apparent that classical theory is unable to explain many things e.g. why a solid is ‘solid’, or why a hot object has the colour that it does. Beyond that, quantum mechanics is needed to explain radioactivity, the chemical properties of matter, how semiconducting devices work, superconductivity, the interaction between light and matter (leading to describing what makes a laser do what it does), the properties of elementary particles such as quarks, muons, neutrinos, Even on the very large scale, quantum effects leave their mark in unexpected ways: the galaxies spread throughout the universe are believed to be macroscopic manifestations of microscopic quantum-induced inhomogeneities present shortly after the birth of the universe, when the universe itself was tinier than an atomic nucleus and almost wholly quantum mechanical. Indeed, the marriage of quantum mechanics – the physics of the very small – with general relativity – the physics of the very large – is believed by some to be the crucial step in formulating a general ‘theory of everything’ that will hopefully contain all the basic laws of nature in one package. The impact of quantum mechanics on our view of the world and the natural laws that govern it, cannot be underestimated.

But the subject is not entirely esoteric. Its consequences have been exploited in many ways that have an immediate impact on the quality of our lives. It has been estimated that the economical impact of quantum mechanics cannot be ignored: about 30% of the gross national product of the United States is based on inventions made possible by quantum mechanics, mostly solid-state devices. If anyone aims to have anything like a broad understanding of the sciences that underpin modern technology, as well as obtaining some insight into the modern view of the character of the physical world, then some knowledge and understanding of quantum mechanics is mandatory. In the broader community, the peculiar things that quantum mechanics says about the way the world works has meant that general interest books on quantum mechanics and related subjects continue to popular with laypersons. This is clear evidence that the community at large and not just the scientific and technological community are very interested in what quantum mechanics has to say. Note that even the term ‘quantum’ has entered the vernacular – it is the name of a car!! The phrase ‘quantum jump’ or ‘quantum leap’ is now in common usage, and incorrectly too: a quantum jump is usually understood to represent a substantial change whereas a quantum jump in its physics context is actually something that is very small.

As we have just seen, quantum mechanics is essential in providing a framework of physical and mathematical principles with which we attempt to understand the physical nature of the world in which we live. Its success in doing just that has been extraordinary. Following the principles of quantum mechanics, it is possible to provide an explanation of everything from the state of the universe immediately after the big bang, to the structure of DNA, to the colour of your socks. Yet for all of that, and in spite of the fact that the theory is now roughly 100 years old, if Planck’s theory of black body radiation is taken as being the birth of quantum mechanics, it is as true now as it was then that no one truly understands the theory, though in recent times, a greater awareness has developed of what quantum mechanics is all about: as well as being a physical theory, it is also a theory of information, that is, it is a theory concerning what information we can gain about the world about us – nature places limitations on what we can ‘know’ about the physical world, but it also gives us greater freedoms concerning what we can do with this ‘quantum information’ (as compared to what we could expect classically), as realized by recent developments in quantum computation, quantum teleportation, quantum cryptography and so on. For instance, quantum cryptography is already a functioning technology – it presents a means by which it is *always* possible to know if there is an eavesdropper listening in on what is supposed to be a secure communication channel. Billions of dollars are

being invested world-wide into research into quantum computing, including Australia, and including Macquarie University Physics Department. Amongst other things, if quantum computing ever becomes realizable, then all security protocols used by banks, defence, and businesses can be cracked. But even if this goal is never reached, trying to achieve it has meant an explosion in our understanding of the quantum information aspects of quantum mechanics, and which may perhaps one day finally lead us to a full understanding of quantum mechanics itself.

1.1 Wave Mechanics vs Quantum Mechanics

Quantum mechanics takes many superficially different mathematical forms called representations, but at its heart it is a mathematically abstract subject expressed in terms of the language of complex linear vector spaces – in other words, linear algebra. In fact, it was in this form that quantum mechanics was first worked out, by Werner Heisenberg, in the 1920s who showed how to represent the physically observable properties of systems as matrices. But not long after, a second version of quantum mechanics appeared, that due to Erwin Schrödinger. Instead of being expressed in terms of matrices and vectors, it was written down in the terms of waves propagating through space and time (at least for a single particle system). These waves were represented by the so-called wave function $\Psi(x, t)$, and the equation that determined the wave function in any given circumstance was known as the Schrödinger equation.

This version of the quantum theory was, and still is, called ‘wave mechanics’. It is fully equivalent to Heisenberg’s version, but because it is expressed in terms of the then more familiar mathematical language of functions and wave equations, and as it was usually far easier to solve Schrödinger’s equation than it was to work with (and understand) Heisenberg’s version, it rapidly became ‘the way’ of doing quantum mechanics, and stayed that way for most of the rest of the 20th century. Its most usual application, built around the wave function Ψ and the interpretation of $|\Psi|^2$ as giving the probability of finding a particle in some region in space, is to describing the structure of matter at the atomic level where the positions of the particles is important, such as in the distribution in space of electrons and nuclei in atomic, molecular and solid state physics. But quantum mechanics is much more than the mechanics of the wave function, and its applicability goes way beyond atomic, molecular or solid state theory. Quantum mechanics is a set of fundamental principles that presumably apply to all physical systems: to the electromagnetic field and the other force fields of nature, to quarks and electrons and other fundamental particles, which can be created or destroyed and which possess such properties as spin, charge, colour, flavour, to many particle systems such as the electrons in a metal, or photons in a laser beam. To describe the quantum properties of such a wide variety of physical phenomena, and to provide a language that contains all of the basic principles of quantum mechanics without being tied to the notion of the wave function, a more general set of principles is required. These principles embody fundamental physical and philosophical issues that can be abstracted from and studied independently of any physical system that could potentially display them. It is these principles abstracted in this way that constitute quantum mechanics. Wave mechanics is then but one mathematical manifestation or representation of this underlying, more general theory. The language of this more general theory is the language of vector spaces, of state vectors and of Hermitean operators and observables, of eigenvalues and eigenvectors, of linear superpositions of states, of time development operators, and so on. As the subject has matured in the latter decades of the 20th century, and with the development of the ‘quantum information’ interpretation of

quantum mechanics, more and more the tendency is to move away from wave mechanics to the more abstract linear algebra version, chiefly expressed in the notation due to Dirac. In doing so, more of the underlying character and physical content of the subject is revealed without the tunnel vision effect that clinging to wave mechanics tends to bring. This deeper perspective is the subject of PHYS301 and PHYS304.

However, in PHYS201, we will be concentrating on wave mechanics. The starting point will be a quick review of the history of quantum mechanics, with the aim of summarizing the essence of the wave mechanical point of view. Following this, a study will be made of the one experiment that is supposed to embody all of the mystery of quantum mechanics – the double slit interference experiment. Amongst other things, this experiment enables us to provide a physical interpretation of the wave function, from which we are then able to develop some of its properties. After ‘deriving’ Schrödinger’s equation, some applications of this equation are described.

The overall approach adopted here is one of inductive reasoning, that is the subject is developed by a process of trying to see what might work, or what meaning might be given to a certain mathematical or physical result or observation, and then testing the proposal against the scientific evidence. The procedure is not a totally logical one, but the result *is* a logical edifice that is only logical after the fact, i.e. the justification of what is proposed is based purely on its ability to agree with what is known about the physical world.