PHYS201

DIffraction and Interference

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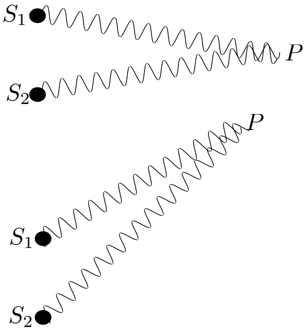
Simple Harmonic Motion

- Oscillatory motion (the simple harmonic oscillator) is found throughout the physical world:
  - Mass on a spring.
  - Waves on a string.
  - Electromagnetic waves are dynamically the same as the simple harmonic oscillator.
    - Important in formulating the quantum version of electromagnetism: photons
    - Elementary particles known as bosons are harmonic oscillators in disguise!!
- These are all essentially examples of *mechanical* oscillations
- But oscillatory properties of *all* waves — sound waves, water waves, light waves, probability (amplitude) waves of quantum mechanics — has other important consequences:
  - Interference and
  - Diffraction

The significance of interference and diffraction

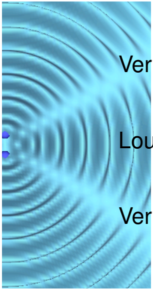
- Interference and diffraction are uniquely characteristic of wave motion:
  - Young's interference experiment showed that light was a form of wave motion
  - Whereas Newton thought that light was made up of 'corpuscles'.
  - Ironically, modern quantum mechanics says that light *is* made up of 'corpuscles', called photons!
- Overlapping waves from two sources combine to produce an *interference pattern*.
  - The separation of the sources d is half the wavelength λ of the waves.
  - Note the regions where the waves cancel (the diagonal lines) – destructive interference.
  - Less easy to see: waves enhance midway between the cancellation regions – constructive interference.

Constructive and destructive interference.



- Waves from the two sources S_1 and S_2 arrive at P 'in-step' and hence reinforce.
  - Waves are said to be 'in phase' and we get constructive interference.
- Waves from the two sources S_1 and S_2 arrive at P 'out-of-step' and hence cancel.
  - Waves are said to be 'out of phase' and we get destructive interference.
- In between talk about 'partial interference'.

Definition of Interference and Diffraction



Two loud speakers

Very quiet

Loud volume

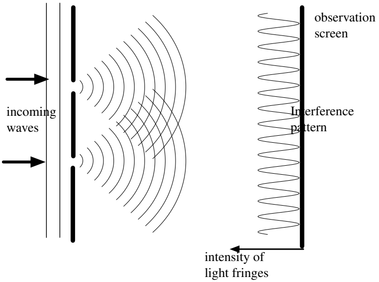
Very quiet

- **Interference** occurs when waves from a *finite* number of sources are simultaneously present (superimposed) in the same region of space.
  - Here sound waves from two speakers emitting a single tone (e.g. middle C 278Hz, $\lambda = 1.2$m) overlap creating regions of loudness (constructive interference) and quiet (destructive interference).
- **Diffraction** is the limiting case of interference when waves from an — essentially — infinite number of sources are superimposed.
  - Diffraction is usually thought of in terms of the spreading of waves as they pass through a narrow opening or bend around an obstacle.
  - Here ocean waves are diffracting around a headland.

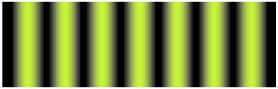
Vector and Scalar theory of interference and diffraction

- Waves will often have an amplitude of oscillation, but also have a direction of oscillation.
  - Waves on a string — just waggle the string in a circular motion.
  - In electromagnetic waves, the electric and magnetic fields are *vectors*: they have both a magnitude and a direction.
  - Have to use vector addition when combining different waves together: a much more difficult calculation
  - We shall assume that the waves are *scalars* — no need for vector addition, just positive and negative numbers (and later, complex numbers).
  - It turns out that the vector and the scalar theories give the same result in many cases!!

Interference



- Interference can arise in a number of ways:
  - Two or more separate sources (e.g. two lasers, two or more radio aerials, two or more loudspeakers) radiating waves that will interfere when they are superimposed.
  - Interference arising from the *division of a wavefront*. e.g. in two slit (Young's) interference experiment.
  - Lower figure on left is of the observed *interference pattern*.

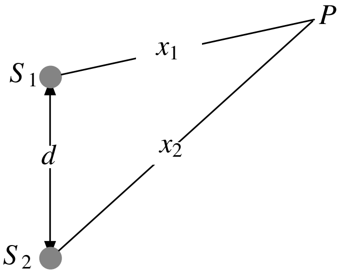


- Also get interference from a wave being partially reflected and partially transmitted through e.g. a sheet of glass.

Interference from two sources

A mathematical description

- Shall study two source interference case in order to introduce
  - constructive interference & destructive interference
  - the importance of phase difference

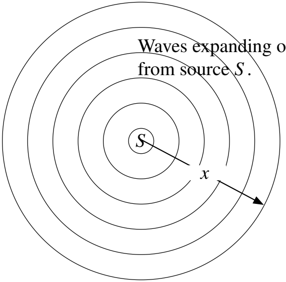


- What is required is the total disturbance at P due to waves from the two sources S_1 and S_2.
  - The 'disturbance' can be any kind of *linear* wave — water wave, sound wave, light wave, gravitational wave, probability amplitude wave . . . but not a shock wave: they are non-linear.
  - Linear waves can be simply added together or 'superimposed'.
  - Shall assume the waves generated by each source will have the same frequency (and wavelength)

Interference from two sources

Description of a single source.

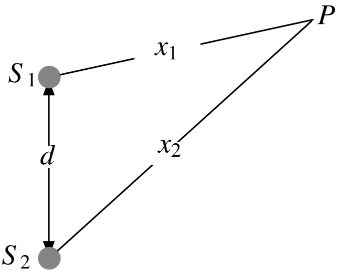
- For an outwardly spherically expanding wave $y = a\sin(\omega t - kx + \phi)$
  - y is called the *amplitude* of the disturbance — note new meaning for ‘amplitude’.
  - a is also called the amplitude so beware of the context.
  - ω (angular frequency) $= 2\pi f$ $\quad k$ (wave number) $= 2\pi/\lambda$
  - At the source $(x = 0)$ $y \propto \sin(\omega t + \phi)$. ϕ is the phase of the source oscillations.
  - Why the proportionality sign?



- the amplitude a falls off as $1/x$.
  - So, at the source, $a = \infty$!!!
  - But no source is a true point, so a will be assumed finite always.
- In fact, we will assume a is a constant.

Interference from two sources

A mathematical description continued



- Amplitude of wave at P due to wave from S_1 is

$$y_1(P) = a_1 \sin(\omega t - kx_1 + \phi_1)$$

- Similarly, the amplitude of the wave at P due to waves originating from S_2 is

$$y_2(P) = a_2 \sin(\omega t - kx_2 + \phi_2)$$

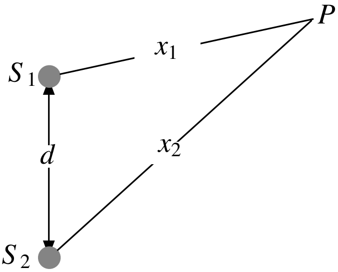
- At the sources $x_1 = 0$ and $x_2 = 0$ the amplitudes are

$$y_1 \propto \sin(\omega t + \phi_1) \quad \text{and} \quad y_2 \propto \sin(\omega t + \phi_2).$$

  - The phase difference $\phi_1 - \phi_2$ tell us by how much the waves at the sources are out-of-step.

Interference from two sources

Mathematical description continued



- The total amplitude $y(P)$ at P is obtained by simple addition: $y(P) = y_1(P) + y_2(P)$
  - This is what we mean by 'superposition' of two waves.

$$y(P) = a_1 \sin(\omega t - kx_1 + \phi_1) + a_2 \sin(\omega t - kx_2 + \phi_2).$$

- What is almost always measured for any wave is not its amplitude but its *intensity*
  - Intensity is defined differently for different kinds of waves, but in every case, it is proportional to the *square of the amplitude*:

$$\text{'Instantaneous Intensity'} \quad I = y^2$$

.

  - This is known as the **instantaneous intensity** as it gives the intensity at each instant in time.

Interference from two sources

Instantaneous intensity

- Instantaneous intensity for combined waves at P is (leave out P for the present)

$$\begin{aligned} I = y^2 &= (y_1 + y_2)^2 \\ &= y_1^2 + y_2^2 + 2y_1 y_2 \\ &= I_1 + I_2 + 2y_1 y_2 \end{aligned}$$

  Here I_1 and I_2 are the instantaneous intensities at P due to waves originating from sources S_1 and S_2 respectively.

- Written out in full:

$$I(P) = a_1^2 \sin^2(\omega t - kx_1 + \phi_1) + a_2^2 \sin^2(\omega t - kx_2 + \phi_2) + 2a_1 a_2 \sin(\omega t - kx_1 + \phi_1)\sin(\omega t - kx_2 + \phi_2)$$

- Now we use trigonometry to work out the last term:

$$\sin A \sin B = \tfrac{1}{2}\left(\cos(A - B) - \cos(A + B)\right).$$

  to give

$$\begin{aligned} I(P) = &a_1^2 \sin^2(\omega t - kx_1 + \phi_1) + a_2^2 \sin^2(\omega t - kx_2 + \phi_2) \\ &+ a_1 a_2 \left\{\cos[k(x_2 - x_1) + \phi_1 - \phi_2] - \cos[k(x_1 + x_2) - 2\omega t - \phi_1 - \phi_2]\right\} \end{aligned}$$

Interference from two sources

Time averaged intensity

- The expression for $I(P)$ just obtained

$$\begin{aligned} I(P) = & a_1^2 \sin^2(\omega t - kx_1 + \phi_1) + a_2^2 \sin^2(\omega t - kx_2 + \phi_2) \\ & + a_1 a_2 \Big\{ \cos[k(x_2 - x_1) + \phi_1 - \phi_2] - \cos[k(x_1 + x_2) - 2\omega t - \phi_1 - \phi_2] \Big\} \end{aligned}$$

  contains terms that are oscillating in time with a frequency 2ω.

- In general however, these oscillations occur so quickly that it is impossible to follow them
  - e.g. for light at optical frequencies $f \sim 10^{15}$ Hz.
  - Even the oscillations of audible sound waves for which $f \sim 200 - 400$ Hz or higher.
  - But for slowly oscillating quantities, like the tide, the amplitude can be monitored directly.
- We shall assume we are working with high frequency signals. In such cases, all we can reasonably measure is the intensity averaged over many periods of oscillation.

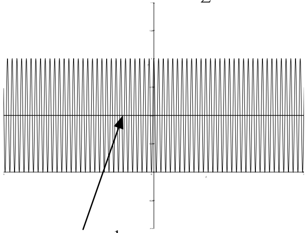
Interference from two sources

Time averaged intensity continued

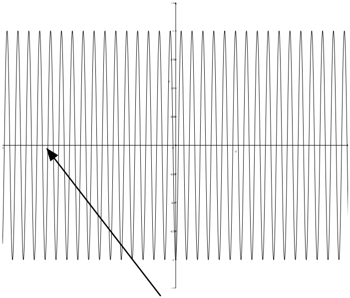
- Shall use some well known results:

$$\overline{\sin^2(\omega t + \theta)} = \frac{1}{2}.$$

$$\overline{\sin(\omega t + \theta)} = 0.$$



Average = $\frac{1}{2}$



Average = 0

- So $\quad I_1 = \overline{y_1^2} = a_1^2 \sin^2(\omega t - kx_1 + \phi_1) \longrightarrow \bar{I}_1 = a_1^2\, \overline{\sin^2(\omega t - kx_1 + \phi_1)} = \frac{1}{2} a_1^2$
- And $\quad \overline{\cos[k(x_1 + x_2) - 2\omega t - \phi_1 - \phi_2]} = 0$

Interference from two sources

Time averaged intensity continued

- Putting it all together

$$\begin{aligned} I(P) =& a_1^2 \sin^2(\omega t - kx_1 + \phi_1) + a_2^2 \sin^2(\omega t - kx_2 + \phi_2) \\ & + a_1 a_2 \Big\{ \cos[k(x_2 - x_1) + \phi_1 - \phi_2] - \cos[k(x_1 + x_2) - 2\omega t - \phi_1 - \phi_2] \Big\} \end{aligned}$$

  becomes

$$\begin{aligned} \bar{I}(P) =& \tfrac{1}{2} a_1^2 + \tfrac{1}{2} a_2^2 + a_1 a_2 \cos[k(x_2 - x_1) + \phi_1 - \phi_2] \\ =& \bar{I}_1 + \bar{I}_2 + 2\sqrt{\bar{I}_1 \bar{I}_2} \cos\delta \end{aligned}$$

  where $\delta = k(x_2 - x_1) + \phi_1 - \phi_2$.

- We shall make two further assumptions:
  - The sources are of equal strength, $a_1 = a_2 = a$ so $\bar{I}_1 = \bar{I}_2 = \bar{I}_0$
  - The sources are in phase $\phi_1 = \phi_2$.
  - Tutorial exercises will look at what happens if these conditions are not satisfied.

Interference from two equal strength sources

Constructive interference

- For equal strength, in phase sources, we get

$$\bar{I}(P) = 2\bar{I}_0 + 2\bar{I}_0 \cos\delta = 2\bar{I}_0(1 + \cos\delta) = 4\bar{I}_0 \cos^2 \tfrac{1}{2}\delta.$$

where
$$\delta = k(x_2 - x_1) = \frac{2\pi}{\lambda}(x_2 - x_1).$$

- **Constructive interference** occurs when the intensity at P reaches a maximum.
  - This will occur when $\cos^2 \frac{1}{2}\delta = 1$:

$$\bar{I}(P) = 4\bar{I}_0 \quad \text{for} \quad \cos^2 \tfrac{1}{2}\delta = 1.$$

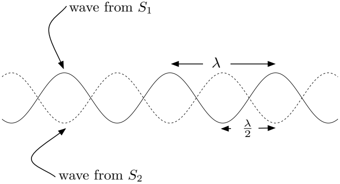
Which gives

$$\tfrac{1}{2}\delta = \frac{\pi}{\lambda}(x_2 - x_1) = n\pi \quad n \quad \text{an integer}$$
$$\Downarrow$$
$$x_2 - x_1 = n\lambda$$

  - The path difference $x_2 - x_1$ must be an integer number of wavelengths.
  - The waves leave S_1 and S_2 in step as $\phi_1 = \phi_2$ and arrive at P in step.

Interference from two equal strength sources
Destructive interference

- **Destructive interference** occurs when $\overline{I}(P) = 4\overline{I}_0 \cos^2 \frac{1}{2}\delta$ is a minimum, i.e. zero.
  - Requires $\cos \frac{1}{2}\delta = 0$ which gives
$$\frac{1}{2}\delta = \frac{\pi}{\lambda}(x_2 - x_1) = (n + \tfrac{1}{2})\pi \quad n \quad \text{an integer}$$
$$\Downarrow$$
$$x_2 - x_1 = (n + \tfrac{1}{2})\lambda.$$
  - The path difference $x_2 - x_1$ must be a half integer number of wavelengths.
  - The waves leave S_1 and S_2 in step, but arrive at P exactly out of step.
  - One wave has to travel $1\frac{1}{2}$ or $2\frac{1}{2}$ or $3\frac{1}{2}\ldots$ wavelengths further on the way to the point P.



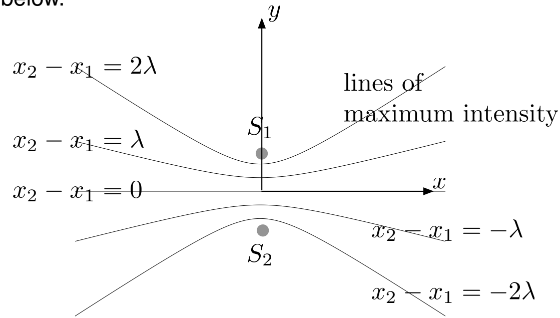
Interference from two equal strength sources

Interference maxima in space

- Can derive a formula for the position in space of the interference maxima:

$$\frac{4y^2}{n^2\lambda^2} - \frac{4x^2}{d^2 - n^2\lambda^2} = 1 \qquad n = 0, 1, 2, 3, \ldots, \quad \text{such that} \quad n\lambda \le d$$

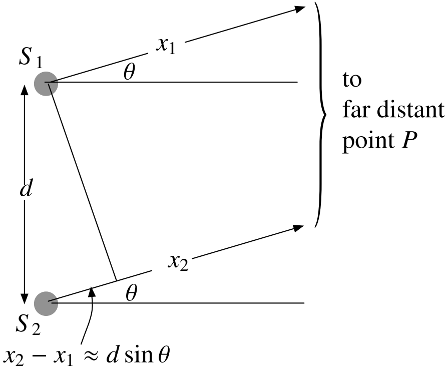
- $d = 3\lambda$ in figure below.



- Will be asked to analyse this result in an assignment question.

Polar Plots

- A very useful way to represent the directional properties of the interference pattern due to two or more sources is to plot the intensity as a function of direction.
- The idea is to work out what the intensity of the combined waves are at a long distance from the sources (the Fraunhofer condition):



- Since $\bar{I}(P) = 4\bar{I}_0 \cos^2 \frac{1}{2}\delta$ with $\delta = \frac{2\pi}{\lambda}(x_2 - x_1)$ then

$$\begin{aligned}\bar{I}(P) =& 4\bar{I}_0 \cos^2\left(\frac{\pi(x_2 - x_1)}{\lambda}\right)\\ =& 4\bar{I}_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)\end{aligned}$$

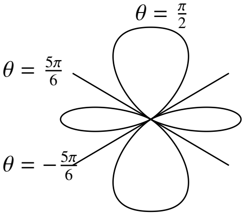
- A natural way to plot this is as a function of angle as a *polar plot*.

Polar Plots continued

- To illustrate, shall suppose that $d = \lambda$. Then

$$\bar{I}(\theta) = 4\bar{I}_0 \cos^2(\pi \sin\theta)$$

- Plot this by calculating $\bar{I}(\theta)$ for each value of θ, but then draw a line from the origin out a distance $\propto \bar{I}(\theta)$ at an angle θ to the 'horizontal' direction.



- It is usually sufficient to determine the angles at which the intensity is a maximum or a minimum, mark those points, and sketch in the curve joining those points.
  - Maxima ($\bar{I} = 4\bar{I}_0$) occur when $\pi\sin\theta = n\pi$, $n = 0, \pm1, \pm2, \ldots$.

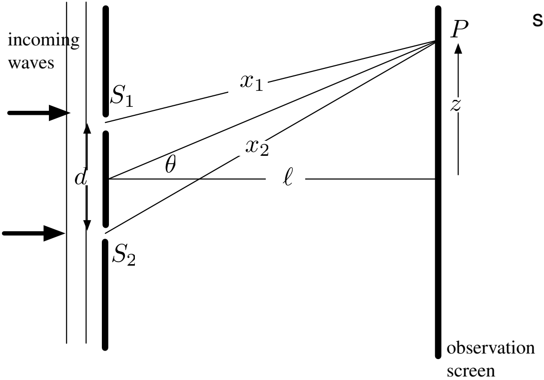
    i.e. $\quad \sin\theta = n \quad n = 0, \pm1, \pm2\ldots$
  - Note that $-1 \leq \sin\theta \leq 1$, which cuts off the allowed values of n to $n = 0, \pm1$.
  - So maxima occur at

    $$\sin\theta = 0 \Rightarrow \theta = 0, \pi \qquad \text{and} \qquad \sin\theta = \pm1 \Rightarrow \theta = \pm\frac{\pi}{2}.$$
  - Minima ($\bar{I} = 0$) occur when $\pi\sin\theta = (n + \frac{1}{2})\pi$, $n = 0, \pm1, \pm2, \ldots$

    i.e. $\quad \sin\theta = (n + \frac{1}{2}) \quad n = 0, \pm1, \pm2, \ldots$
  - So minima occur at $\quad \sin\theta = \pm\frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$.

Young's interference experiment

- This was the first experiment (1803) to show that light was a form of wave motion, and not made up of 'bullet-like' corpuscles as proposed by Newton.



- Waves are incident from a very far distant source.
  - The 'wave fronts' reach the slits S_1 and S_2 simultaneously, so waves are in phase when they reach the slits.
  - The waves spread out after passing through the slits (diffraction), so the slits act as in phase sources of waves.
  - Shall assume the Fraunhofer condition

$$\ell \gg d$$

so the approximation can be made:

$$x_2 - x_1 \approx d \sin\theta$$

Young's interference experiment continued

- The set-up is equivalent to the two source problem studied earlier, so

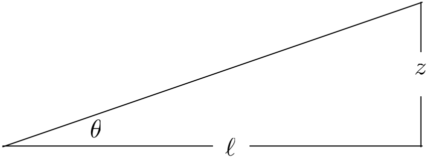
$$\bar{I}(P) = 4\bar{I}_0 \cos^2 \tfrac{1}{2}\delta \qquad \delta = \frac{2\pi}{\lambda}(x_2 - x_1)$$

  where $\bar{I}_0$ is the intensity at P due to the waves from one slit only.

- Using approximate result $x_2 - x_1 \approx d \sin\theta$:

$$\bar{I}(P) = 4\bar{I}_0 \cos^2\left(\frac{\pi d}{\lambda} \sin\theta\right).$$

- We are after the interference pattern on the observation screen, so we want $\bar{I}(P)$ as a function of z.



- For $\theta < 25^\circ$ $\qquad \sin\theta \approx \tan\theta = \dfrac{z}{\ell}$

$$\therefore \qquad \bar{I}(P) \approx 4\bar{I}_0 \cos^2\left(\frac{\pi d}{\lambda}\frac{z}{\ell}\right).$$

Young's interference experiment
Interference fringes

- Maxima when $\cos^2\left(\frac{\pi d}{\lambda}\frac{z}{\ell}\right) = 1$

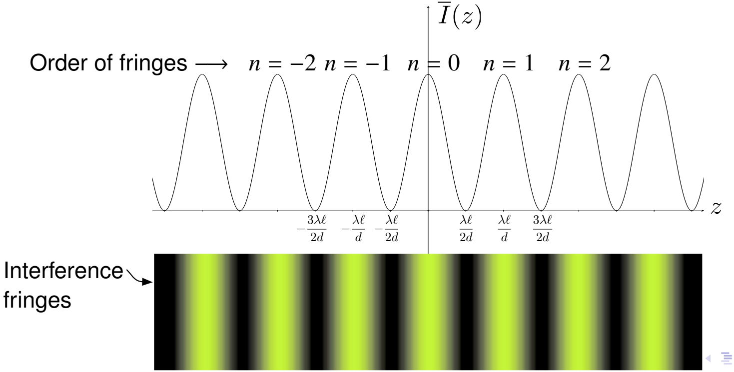
$$\Rightarrow \qquad \frac{\pi d}{\lambda}\frac{z}{\ell} = n\pi \quad n = 0, \pm 1, \pm 2, \ldots.$$

$$\therefore z = n\left(\frac{\lambda \ell}{d}\right)$$

- Minima when $\cos^2\left(\frac{\pi d}{\lambda}\frac{z}{\ell}\right) = 0$

$$\Rightarrow \qquad \frac{\pi d}{\lambda}\frac{z}{\ell} = (n + \tfrac{1}{2})\pi \quad n = 0, \pm 1, \pm 2, \ldots.$$

$$\therefore z = (n + \tfrac{1}{2})\left(\frac{\lambda \ell}{d}\right)$$



Order of fringes ⟶ $n = -2$ $n = -1$ $n = 0$ $n = 1$ $n = 2$

Interference fringes

Young's interference experiment

Interference fringes continued

- Each bright interference maximum is known as an *interference fringe*
- The maximum positioned at $z_n = n\left(\frac{\lambda\ell}{d}\right)$ is known as the n$^{\text{th}}$ order fringe.
- Adjacent fringes are equally separated:

$$z_{n+1} - z_n = \frac{\lambda\ell}{d}$$

(except for angles greater than about 25° when the fringes become further apart.)

- Two limiting cases:
  - For increasing d, the fringes become closer together.
  - For decreasing d, eventually find $\pi\frac{d}{\lambda}\sin\theta << 1.$
    But $\cos x \approx 1$ if $x \ll 1$ so

$$\bar{I}(P) = 4\bar{I}_0\cos^2\left(\frac{\pi d}{\lambda}\sin\theta\right) \approx 4\bar{I}_0.$$

  - Thus there are *no fringes* on the observation screen.
  - Get the result expected if the two sources (slits) coincided.

Some useful properties of complex numbers

- In the following analysis we will need to make use of complex numbers as an aid in adding together large numbers of sin functions.
  - Recall that we have had to calculate sums like
$$y = a_1 \sin(\omega t - kx_1 + \phi_1) + a_2 \sin(\omega t - kx_2 + \phi_2).$$
  - Such sums become prohibitively difficult to do if we have 5, 10, 50, . . . separate sources.
  - Can use complex number methods to greatly simplify such calculations.
- Recall Euler's theorem: $e^{ix} = \cos x + i \sin x \qquad i = \sqrt{-1}.$
  - So that
$$\cos x = \text{Re}\, e^{ix} \qquad \text{Re} \equiv \text{real part}$$
$$\sin x = \text{Im}\, e^{ix} \qquad \text{Im} \equiv \text{imaginary part.}$$
  - and complex conjugate: $e^{-ix} = \cos x - i \sin x.$
  - and $\cos x = \dfrac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}.$

A useful geometric sum

- Will use complex number methods to calculate the sum of N terms:

$$S = \sin b + \sin(b - \delta) + \sin(b - 2\delta) + \ldots + \sin(b - (N-1)\delta)$$

  - arises in study of N-slit interference and diffraction through a slit.

- Impossible to do as is, but can turn it into a simple problem by using complex algebra:

$$\sin(b - n\delta) = \mathrm{Im}\, e^{i(b-n\delta)}$$

  so that

$$\begin{aligned} S =& \mathrm{Im}\left[e^{ib} + e^{i(b-\delta)} + e^{i(b-2\delta)} + \ldots + e^{i(b-(N-1)\delta)}\right] \\ =& \mathrm{Im}\left\{e^{ib}\left[1 + e^{-i\delta} + e^{-2i\delta} + \ldots + e^{-i(N-1)\delta}\right]\right\} \end{aligned}$$

- Now put $r = e^{-i\delta}$. We end up with a geometric series with common ratio r:

$$\begin{aligned} S =& \mathrm{Im}\left\{e^{ib}\left[1 + r + r^2 + \ldots + r^{N-1}\right]\right\} \\ =& \mathrm{Im}\left\{e^{ib}\frac{1 - r^N}{1 - r}\right\} \\ =& \mathrm{Im}\left\{e^{ib}\frac{1 - e^{-iN\delta}}{1 - e^{-i\delta}}\right\} \end{aligned}$$

A useful geometric sum (continued)

- The expression just obtained is expressed in terms of complex quantities. We need a result in terms of real quantities. So, a trick:

$$S = \text{Im}\left\{e^{ib}\frac{1-e^{-iN\delta}}{1-e^{-i\delta}}\right\} = \text{Im}\left\{e^{ib}\cdot\frac{e^{-iN\delta/2}}{e^{-i\delta/2}}\cdot\frac{e^{iN\delta/2}-e^{-iN\delta/2}}{e^{i\delta/2}-e^{-i\delta/2}}\right\}$$

which sets us up to use $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$ to give

$$S = \text{Im}\left\{e^{ib}e^{-i(N-1)\delta/2}\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right\} = \frac{\sin(N\delta/2)}{\sin(\delta/2)}\text{Im}\left\{e^{i(b-(N-1)\delta/2)}\right\}$$

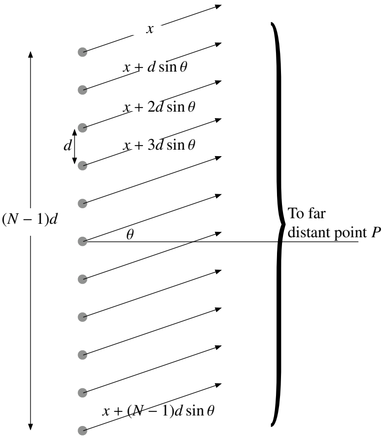
$$\therefore \quad S = \frac{\sin(N\delta/2)}{\sin(\delta/2)}\sin\left[b-(N-1)\delta/2\right].$$

- So finally

$$\sin b + \sin(b-\delta) + \sin(b-2\delta) + \ldots + \sin(b-(N-1)\delta) = \frac{\sin(N\delta/2)}{\sin(\delta/2)}\sin\left[b-(N-1)\delta/2\right]$$

Interference from a linear array of N equal sources

- Shall now generalize to a linear array of N identical sources all radiating in phase at the same frequency.



- Each source will have amplitude a
- Suppose the first source is a distance x from P.
    - Each successive source an extra distance $d\sin\theta$ from P
- The total wave amplitude at P will be

$$\begin{aligned} y(P) =& y_1 + y_2 + \ldots + y_N \\ =& a\sin(\omega t - kx) + a\sin(\omega t - k(x + d\sin\theta)) \\ &+ a\sin(\omega t - k(x + 2d\sin\theta)) + \ldots \\ &+ a\sin(\omega t - k(x + (N-1)d\sin\theta)) \end{aligned}$$

- This is the same kind of sum we evaluated earlier!

Interference from a linear array of N equal sources

Evaluation of amplitude sum

- Have shown that the total amplitude from N sources is:

$$\begin{aligned} y(P) =& a\sin(\omega t - kx) + a\sin(\omega t - k(x + d\sin\theta)) + a\sin(\omega t - k(x + 2d\sin\theta)) + \ldots \\ &+ a\sin(\omega t - k(x + (N-1)d\sin\theta)) \end{aligned}$$

- Have shown earlier that

$$\sin b + \sin(b-\delta) + \sin(b-2\delta) + \ldots + \sin(b-(N-1)\delta) = \frac{\sin(N\delta/2)}{\sin(\delta/2)}\sin\left[b-(N-1)\delta/2\right]$$

- Can now make the identifications:

$$\delta = kd\sin\theta \qquad b = \omega t - kx$$

  and use our formula to give

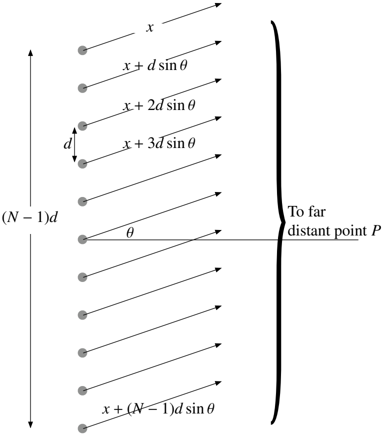
$$y(P) = a\sin(\omega t - kx - (N-1)\delta/2)\cdot\frac{\sin\left(\frac{1}{2}N\delta\right)}{\sin\left(\frac{1}{2}\delta\right)}.$$

- The time averaged intensity is then

$$\bar{I}(P) = \bar{I}_0\left(\frac{\sin\left(\frac{1}{2}N\delta\right)}{\sin\left(\frac{1}{2}\delta\right)}\right)^2 \qquad \text{with} \qquad \bar{I}_0 = \frac{1}{2}a^2 \quad \text{the intensity due to one source.}$$

Interference from a linear array of N equal sources

Checking the formula



- Have shown that the intensity at P is

$$\bar{I}(P) = \bar{I}_0 \left(\frac{\sin\left(\frac{1}{2}N\delta\right)}{\sin\left(\frac{1}{2}\delta\right)} \right)^2$$

  - Check for $N = 2$:

$$\bar{I}(P) = \bar{I}_0 \frac{\sin^2\delta}{\sin^2\frac{1}{2}\delta} = \bar{I}_0 \frac{4\sin^2\frac{1}{2}\delta\cos^2\frac{1}{2}\delta}{\sin^2\frac{1}{2}\delta} = 4\bar{I}_0\cos^2\frac{1}{2}\delta$$

    as before.

  - Can simplify the results for $N = 3, 4$ but gets tough for larger N

- Usually put $\beta = \frac{1}{2}\delta = \frac{\pi d}{\lambda}\sin\theta$ to give

$$\bar{I}(P) = \bar{I}_0 \frac{\sin^2 N\beta}{\sin^2\beta}$$

Interference from a linear array of N equal sources
Structure of interference pattern

- **Principal maxima**
  - A maximum will occur if all the waves arrive at P *exactly in phase*.
  - This can occur here if $d\sin\theta = n\lambda$
    - the distance from one source to P will be a *whole number of wavelengths* more (or less) than its neighbour (or any other source).
  - The condition for a maximum is then $\beta = n\pi$ but the maximum intensity is indeterminate:

$$\bar{I}_{\text{prin. max.}} = \bar{I}_0 \frac{\sin^2 nN\pi}{\sin^2 n\pi} = \frac{0}{0}!!!$$

  - We have to calculate this by taking a limit. Put $\beta = n\pi + \epsilon$:

$$\bar{I}_{\text{prin. max.}} = \bar{I}_0 \frac{\sin^2(nN\pi + N\epsilon)}{\sin^2(n\pi + \epsilon)} = \bar{I}_0 \frac{\sin^2 N\epsilon}{\sin^2 \epsilon} = \bar{I}_0 \left[\frac{\sin N\epsilon}{N\epsilon} \cdot \frac{\epsilon}{\sin\epsilon} \right]^2 \cdot N^2$$

and take the limit as $\epsilon \to 0$, using

$$\lim_{\epsilon\to 0} \frac{\sin\epsilon}{\epsilon} = 1$$

to give $\bar{I}_{\text{prin. max.}} = \bar{I}_0 N^2$ for $d\sin\theta = n\lambda$, the **principal maxima of order** n.

Interference from a linear array of N equal sources

Structure of interference pattern (continued)

- **Minima**
  - Minima will occur when $\bar{I}(P) = 0$, i.e. $\dfrac{\sin N\beta}{\sin\beta} = 0$

    which gives $\sin N\beta = 0$ with $\sin\beta \neq 0$

    - *If both* $\sin N\beta$ and $\sin\beta$ equal zero, get a principal maximum!
  - From $\sin N\beta = 0$ we get $\quad N\beta = n\pi \qquad n = 0, \pm 1, \pm 2, \dots$

    but $\sin\beta \neq 0$ excludes $n = 0, \pm N, \pm 2N \dots$
  - So minima occur at

$$\beta = \frac{n\pi}{N} \quad n = \cancel{0}, \pm 1, \pm 2, \dots, \pm(N-1), \cancel{\pm N}, \pm(N+1), \dots, \pm(2N-1), \cancel{\pm 2N}, \pm(2N+1), \dots$$

  - Or spelt out, for positive β the minima are at:

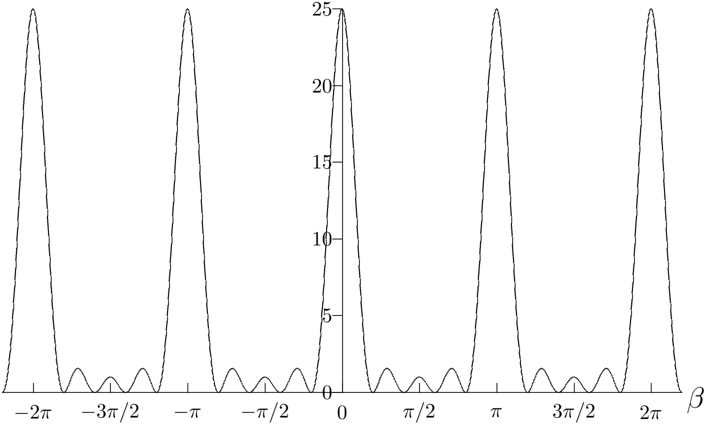
$$\beta = \underset{\underset{\text{principal maximum}}{\uparrow}}{\cancel{0}}, \frac{\pi}{N}, \frac{2\pi}{N}, \dots, \frac{(N-1)\pi}{N}, \underset{\underset{\text{principal maximum}}{\uparrow}}{\cancel{\pi}}, \pi + \frac{\pi}{N}, \pi + \frac{2\pi}{N}, \dots, \pi + \frac{(N-1)\pi}{N}, \quad \text{and so on}$$

and so on.

  - Note that there will be $N - 1$ minima between successive principal maxima.

Interference Pattern for $N = 5$ sources

Interference pattern with $N = 5$ sources



- There are 4 minima between successive principal maxima
- There are 3 subsidiary maxima between successive principal maxima

Interference from a linear array of N equal sources

Structure of interference pattern (continued)

- **Subsidiary maxima**
  - We have determined the position of *principal maxima* where *all* the waves from *all* the sources arrive at P in phase.
  - We have also found that there are $N-1$ minima between these successive principal maxima.
  - So there must be further maxima between these minima!!
  - These lesser maxima occur when there is partial constructive interference.
  - The position and magnitude of these *subsidiary maxima* found using calculus. Thus, setting

    $$\frac{d\bar{I}}{d\beta}=0 \quad \text{with} \quad \bar{I}=\bar{I}_0\left(\frac{\sin N\beta}{\sin\beta}\right)^2$$

    gives

    $$\tan\beta=\frac{\tan N\beta}{N}.$$

  - This is a transcendental equation with no exact solutions.
  - We can find the approximate positions of these maxima by assuming that they lie half-way between successive minima.

Interference from a linear array of N equal sources

Approximate solution for subsidiary maxima.

- The minima occur at

$$\beta = \frac{n\pi}{N} \quad n = \text{integer}, n \neq \text{multiple of } N.$$

- The subsidiary maxima will then occur at approximately

$$\beta = \frac{(n+\frac{1}{2})\pi}{N} \quad n = \text{integer}, n \neq \text{multiple of } N.$$

- These subsidiary maxima lie *between* successive minima.
  - As there are $N - 1$ minima, there will be $N - 2$ subsidiary maxima between successive principal maxima.
  - So there are no subsidiary maxima for $N = 2$.

- The strength of a subsidiary maximum is given by

$$\bar{I}_{\text{sub. max.}} = \bar{I}_0 \left(\frac{\sin\left[N(n+\frac{1}{2})\pi/N\right]}{\sin(n+\frac{1}{2})\pi/N} \right)^2 = \frac{\bar{I}_0}{\sin^2\left[(n+\frac{1}{2})\pi/N\right]}$$

Interference from a linear array of N equal sources

Approximate solution for subsidiary maxima (continued)

- The first subsidiary maximum (i.e. the first subsidiary maximum next to a principal maximum) has a strength given by

$$\bar{I}_{\text{sub. max.}} = \frac{\bar{I}_0}{\sin^2\left(\frac{3}{2}\pi/N\right)} \quad \text{with } n = 1.$$

- For N large we have $\sin\left(\frac{3}{2}\pi/N\right) \approx \left(\frac{3}{2}\pi/N\right)$ so that

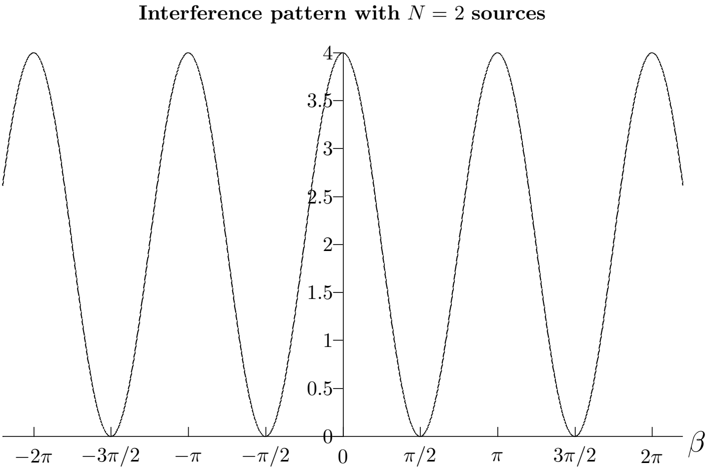
$$\bar{I}_{\text{sub. max.}} \approx \frac{4\bar{I}_0 N^2}{9\pi^2} \approx 0.045\,\bar{I}_0 N^2$$

  - A principal maximum has a strength of $\bar{I}_{\text{prin. max.}} = N^2\bar{I}_0$

- So the first subsidiary maximum is $\sim 4.5\%$ of the principal maximum.

- It is time to put all this together with some examples.

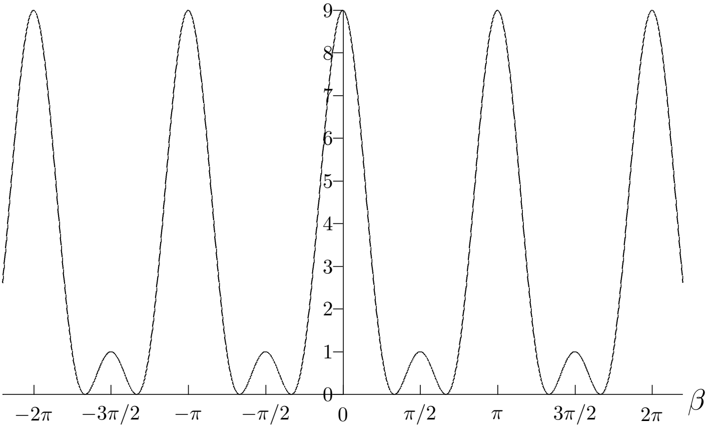
Interference Pattern for $N = 2$ sources



- There is one minimum between successive principal maxima
- There are no subsidiary maxima

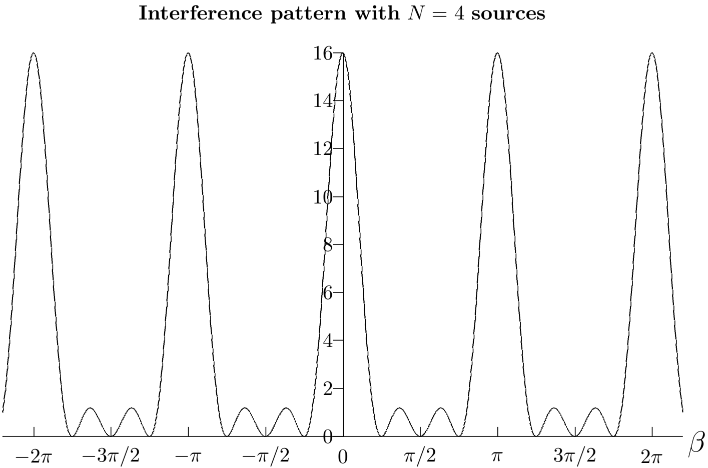
Interference Pattern for $N = 3$ sources

Interference pattern with $N = 3$ sources



- There are 2 minima between successive principal maxima
- There is one subsidiary maximum between successive principal maxima

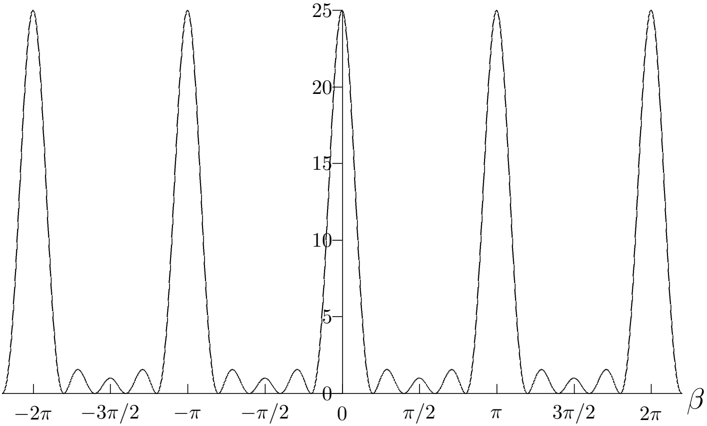
Interference Pattern for $N = 4$ sources



- There are 3 minima between successive principal maxima
- There are 2 subsidiary maxima between successive principal maxima

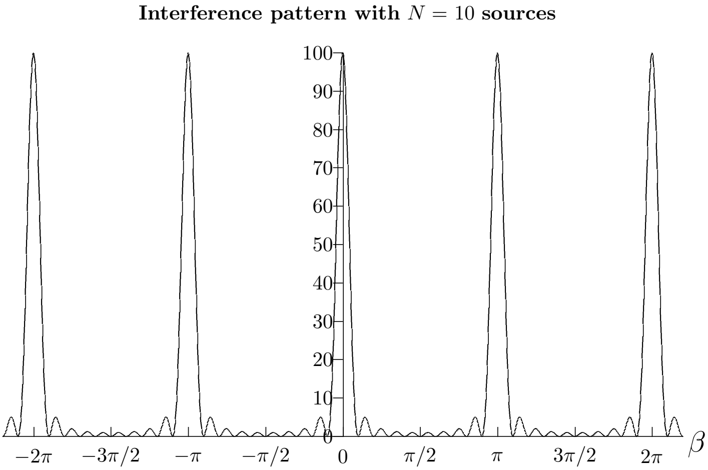
Interference Pattern for N = 5 sources

Interference pattern with $N = 5$ sources



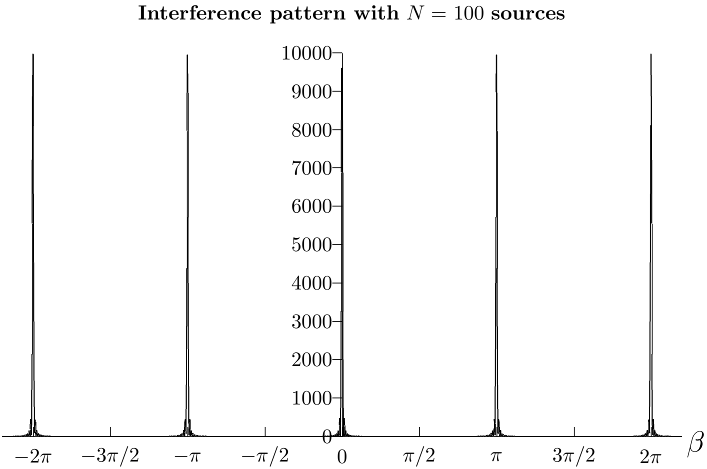
- There are 4 minima between successive principal maxima
- There are 3 subsidiary maxima between successive principal maxima

Interference Pattern for $N = 10$ sources



- There are 9 minima between successive principal maxima
- There are 8 subsidiary maxima between successive principal maxima

Interference Pattern for $N = 100$ sources



- There are 99 minima between successive principal maxima
- There are 98 subsidiary maxima between successive principal maxima

Interference Pattern Polar Plots

- The previous plots of interference patterns where plots as a function of $\beta = \pi d \sin\theta/\lambda$, and does not take account of the restriction that $-1 \leq \sin\theta \leq 1$.
  - This condition has the effect of limiting the number of principal maxima that can occur in practice.
  - For instance, since maxima occur when $d \sin\theta = n\lambda$, the possible values of n, and hence the number of maxima are restricted by

    $$-1 \leq n\lambda/d \leq 1$$

    and hence

    $$-d/\lambda \leq n \leq d/\lambda.$$

- Examples:
  - If $d < \lambda$ then $d/\lambda < 1$ and the only maximum occurs for $n = 0$.
  - If $d = \lambda$ then $-1 \leq n \leq 1$ and there will be three maxima for $n = 0, \pm 1$ i.e.

    $$\sin\theta = 0, \pm 1 \implies \theta = 0, \pm\pi/2.$$

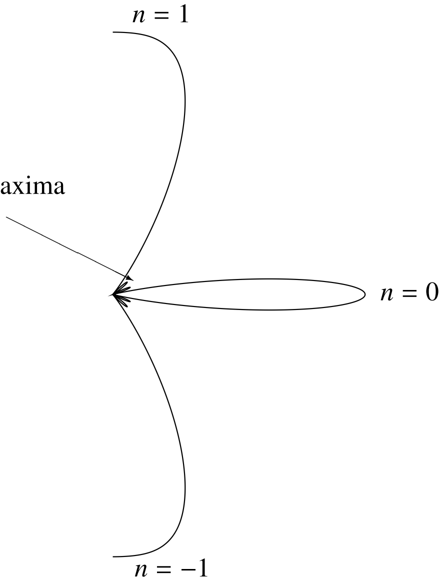
  - If $d = 2.5\lambda$ then $-2.5 \leq n \leq 2.5$ and there will be 5 maxima for $n = 0, \pm 1, \pm 2$ i.e.

    $$\sin\theta = n\lambda/d = 0.4n = 0, \pm 0.4, \pm 0.8 \implies \theta = 0, \pm 0.41 \text{ radians} = 23.6^\circ, \pm 0.93 \text{ radians} = 53^\circ$$

    and so on.

Interference Pattern Polar Plot II

$$\begin{aligned} d/\lambda &= 1 \\ N &= 4 \end{aligned}$$

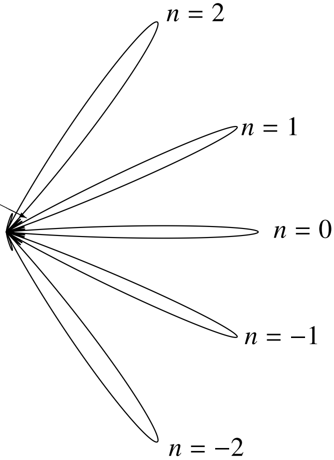


Two subsidiary maxima
hidden between
principal maxima

Interference Pattern Polar Plot III

$$\begin{aligned} d/\lambda &= 2.5 \\ N &= 4 \end{aligned}$$

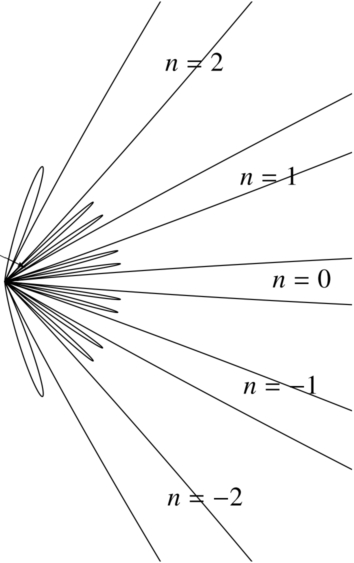
Two subsidiary maxima hidden between principal maxima



Interference Pattern Polar Plot IV

Subsidiary Maxima

$$\begin{aligned} d/\lambda &= 2.5 \\ N &= 4 \end{aligned}$$

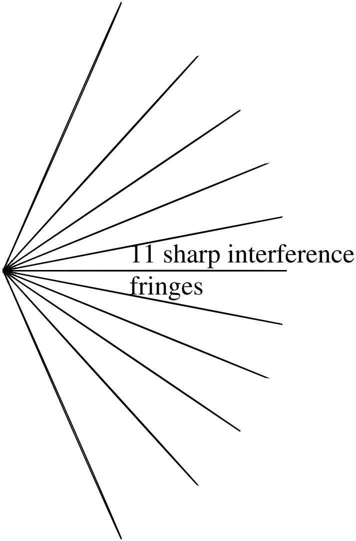


Two subsidiary maxima
hidden between
principal maxima

Interference Pattern Polar Plot V

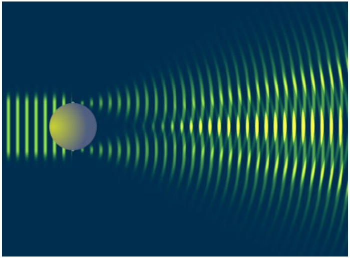
Diffraction grating

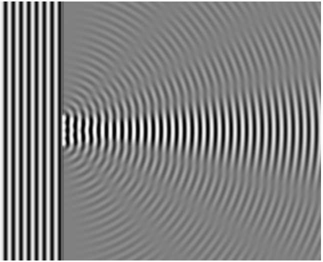
$$\begin{aligned} d/\lambda &= 5.5 \\ N &= 40 \end{aligned}$$



Diffraction

- Diffraction is usually understood as the phenomenon in which waves ‘bend’ around obstacles and around corners.





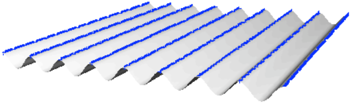
- Diffraction can be understood as the limiting case of interference, but due to the interference of waves from an infinity of sources.

- What are these ‘sources’ and how do they enable us to understand diffraction?
    - The ‘sources’ are all the points on a wavefont. Each such source radiates so-called Huygen’s wavelets, and it is these wavelets that combine to produce the propagating wavefront.
    - But first, what is a wave front?

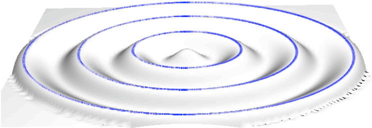
Wavefronts

- In general, a wavefront is those parts of a wave that are at the same phase in its oscillation.
  - A simple example is the ‘crest of a wave’: everywhere that the wave has reached its maximum amplitude

- Wavefronts of plane waves are a set of parallel lines (in 2D):



- Wavefronts of circular waves are a set of concentric circles (in 2D):



- But a wave front is not just the crest of a wave.

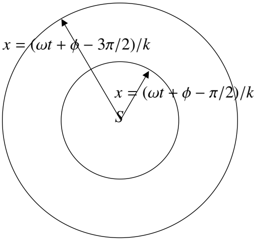
WaveFronts (continued)

- *Mathematically* the definition of a wave front is all to do with phase
- For the wave amplitude produced by a point source: $y = a\sin(\omega t - kx + \phi)$.
  - The whole quantity $(\omega t - kx + \phi)$ is known as the *phase* of the wave.
    - (Unfortunately, ϕ is also sometimes referred to as the phase, so beware.)
- In general, a wavefront is a surface for which the phase has a *constant value*.
  - For example, 'the crest of a wave' is where the wave has its maximum amplitude a $\omega t - kx + \phi = (n + \frac{1}{2})\pi$.

- Now'freeze' the wave at some instant in time t.
  - The points where y has a maximum value will be those a distance x from the source, given by

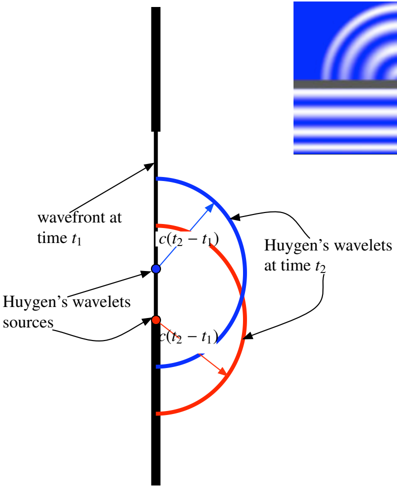
    $$x = \left(\omega t + \phi - (n + \tfrac{1}{2})\pi\right)/k \quad n \text{ an integer.}$$

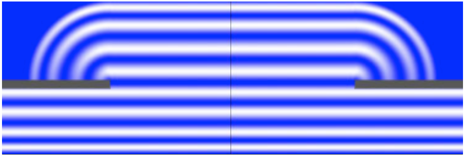
  - This is just the equation for circles (or spheres in three dimensions) centered on S.
  - These circles are examples of *wavefronts*.
  - If n not an integer, still get a wavefront, but not the crest of a wave.



WaveFronts and Huygen's wavelets

- Wavefronts for plane waves passing through a slit *spread out* as they pass through the opening:

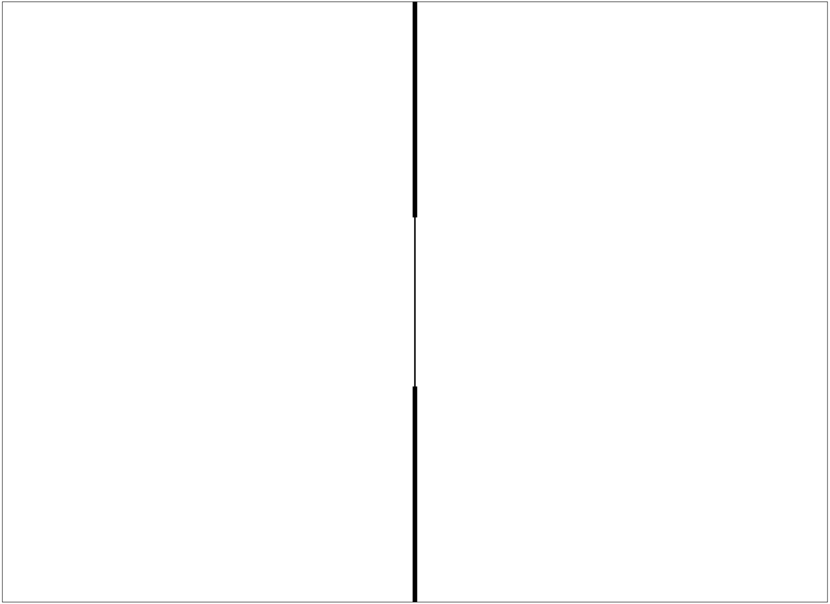




- Can determine how are wave front moves through space by use of the Huygens-Fresnel construction
  - Suppose that *each point on a wavefront acts as a source of spherical wavelets (called Huygen's wavelets)*:
    - These wavelets have the same frequency as the primary wave
    - They have the same phase at their source as the primary wave
    - They propagate at the same velocity as the primary wave.

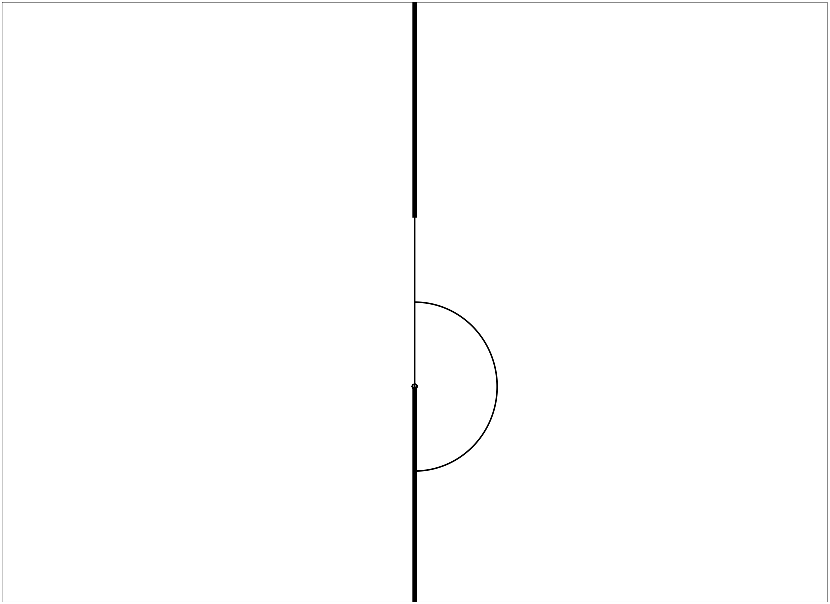
Wavefront propagation via Huygen-Fresnel construction

Initial wavefront



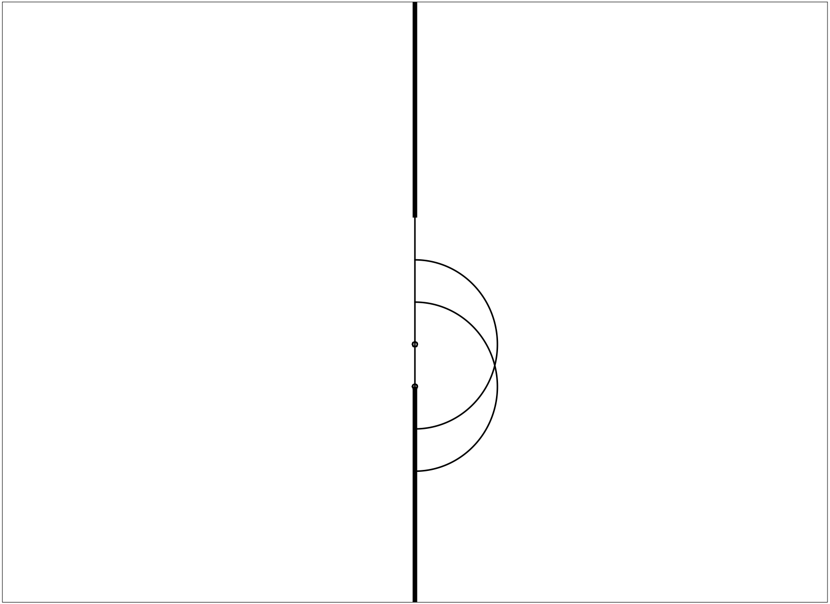
Wavefront propagation via Huygen-Fresnel construction

Sources of Huygen's wavelets



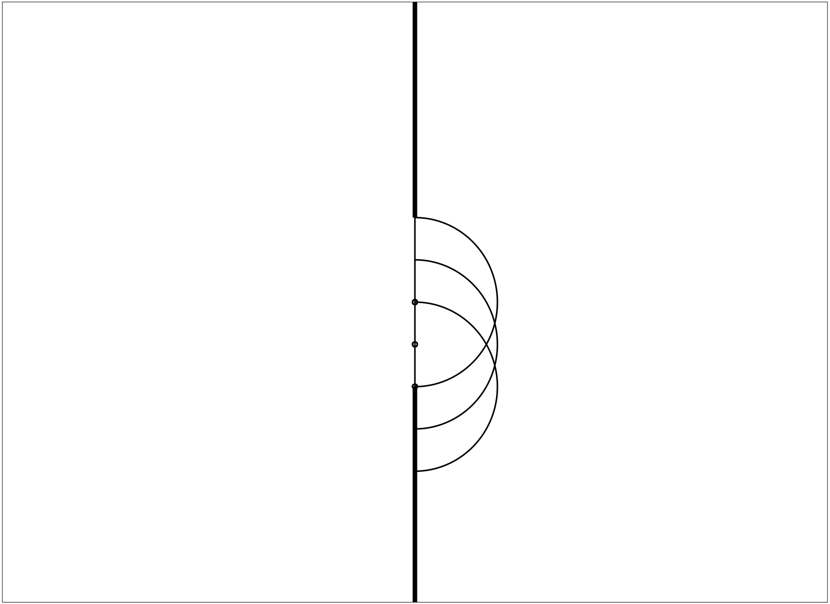
Wavefront propagation via Huygen-Fresnel construction

Adding in the wavelets from each source



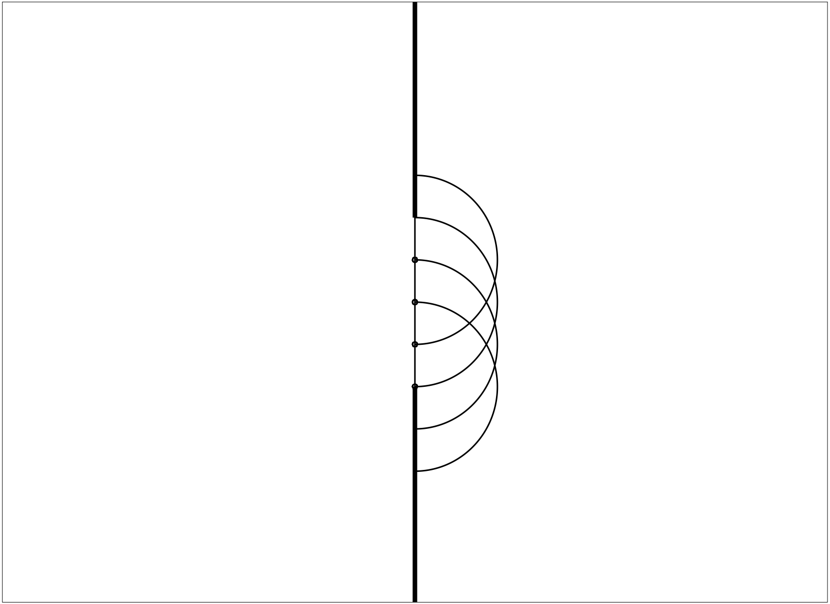
Wavefront propagation via Huygen-Fresnel construction

Adding in the wavelets from each source



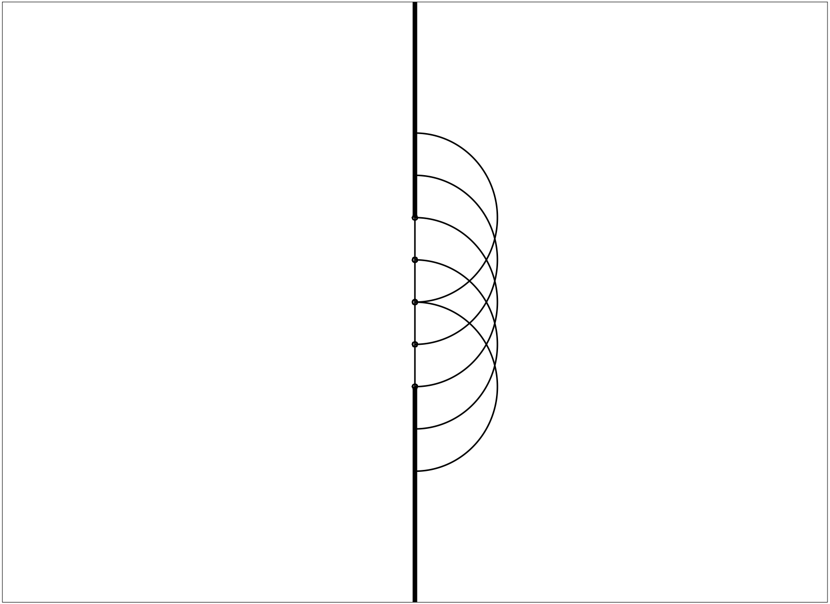
Wavefront propagation via Huygen-Fresnel construction

Adding in the wavelets from each source



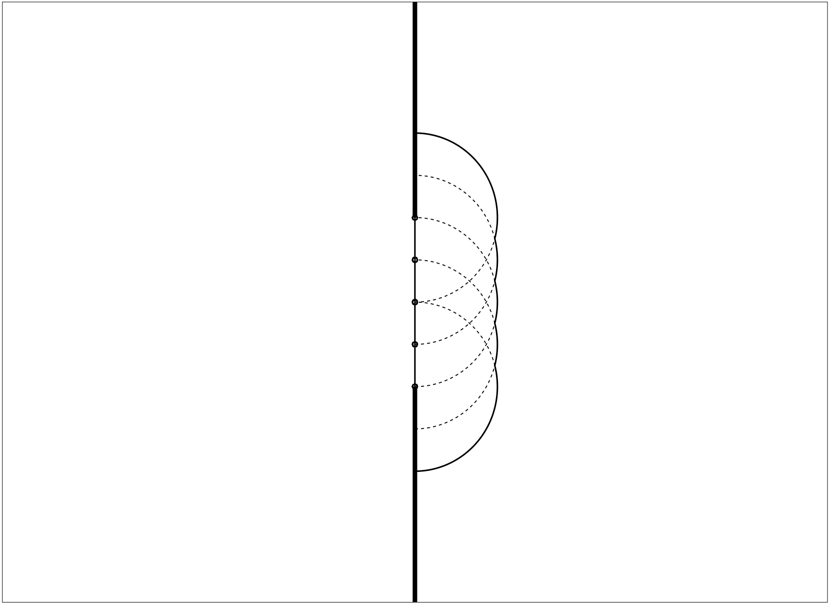
Wavefront propagation via Huygen-Fresnel construction

Adding in the wavelets from each source



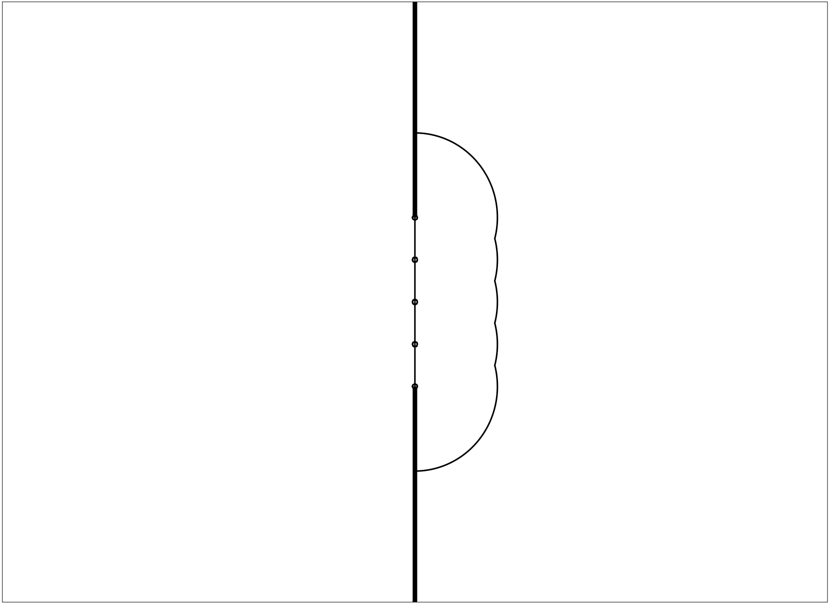
Wavefront propagation via Huygen-Fresnel construction

Leading edge of all the wavelets



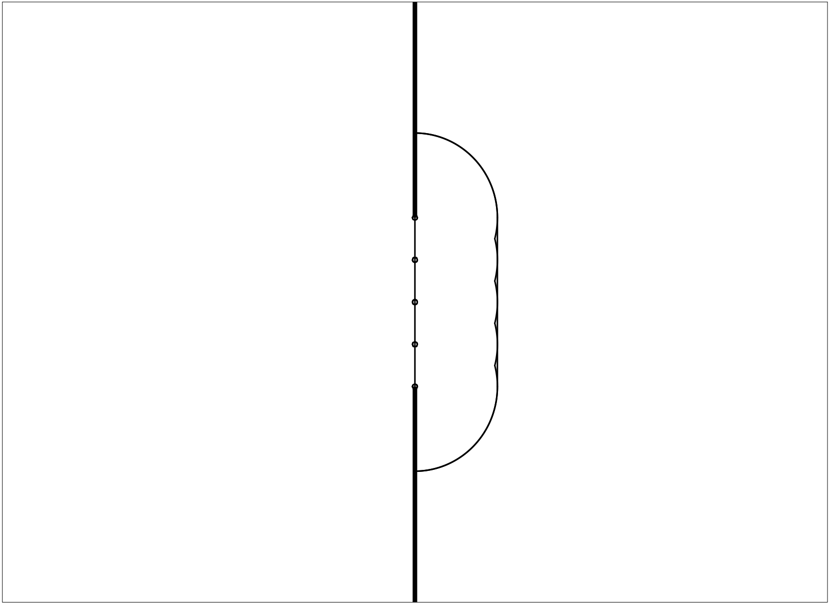
Wavefront propagation via Huygen-Fresnel construction

Approximate form of new wavefront



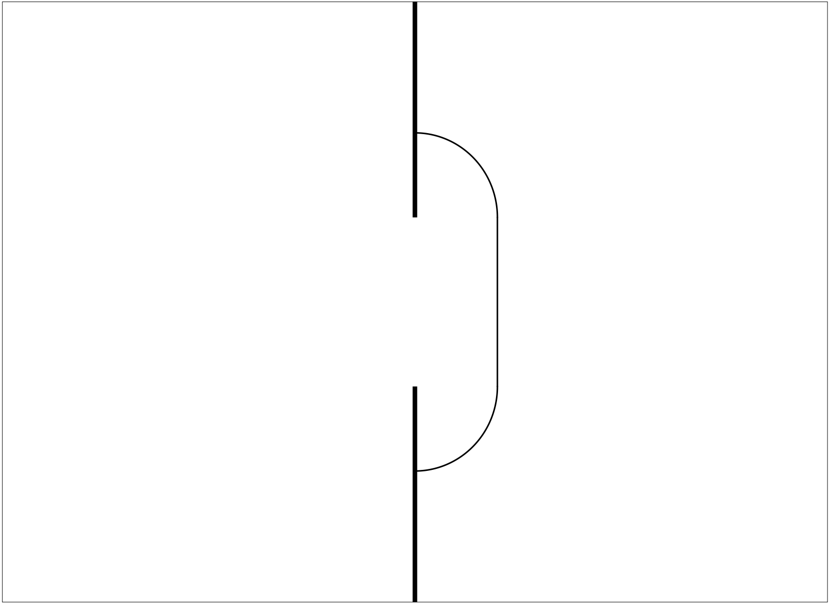
Wavefront propagation via Huygen-Fresnel construction

Fitting the new wavefront



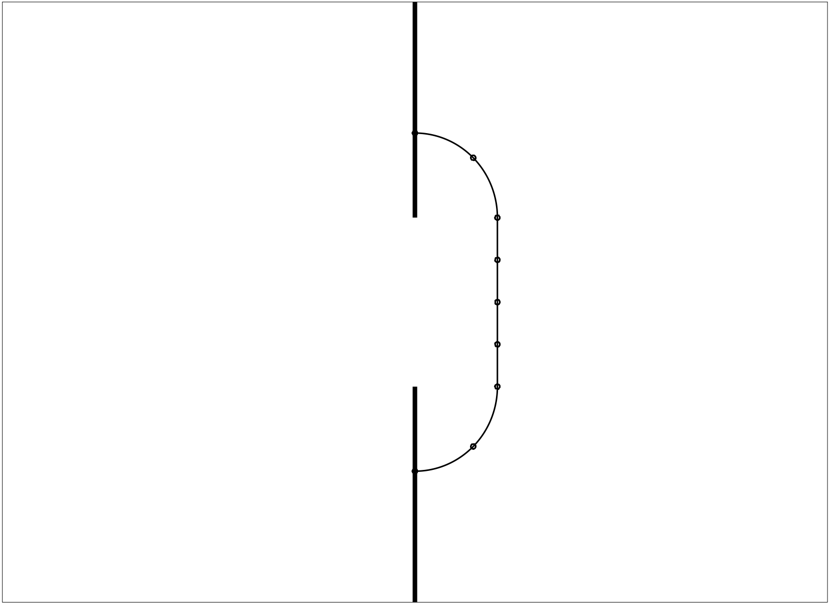
Wavefront propagation via Huygen-Fresnel construction

The new wavefront at last!



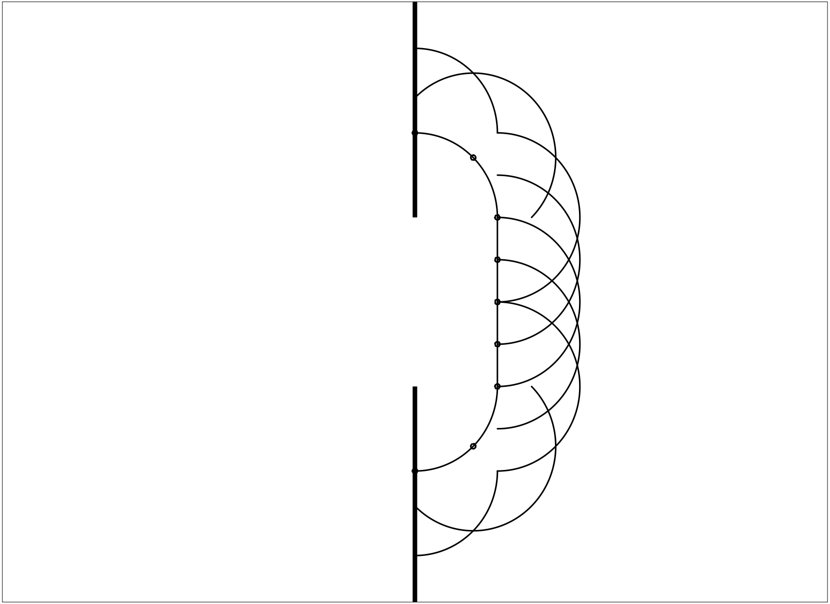
Wavefront propagation via Huygen-Fresnel construction

Huygen's wavelets sources



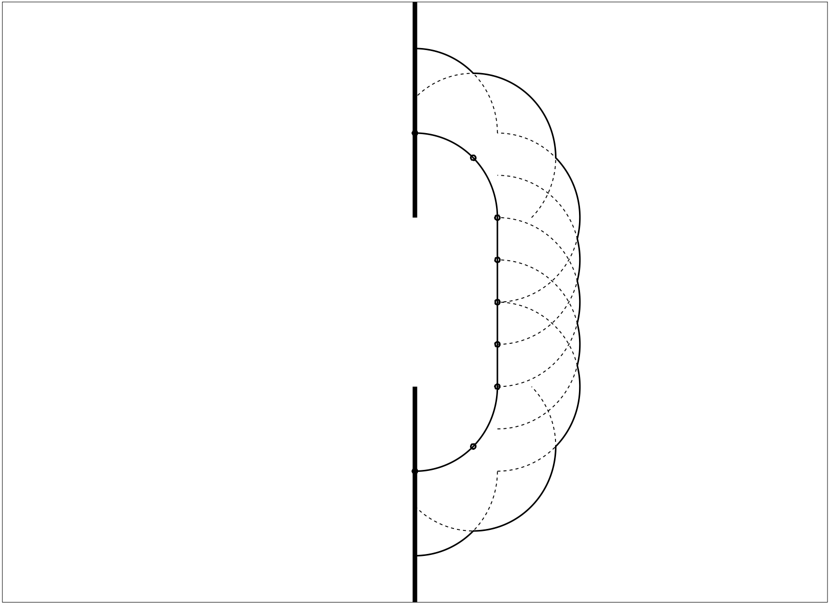
Wavefront propagation via Huygen-Fresnel construction

The wavelets



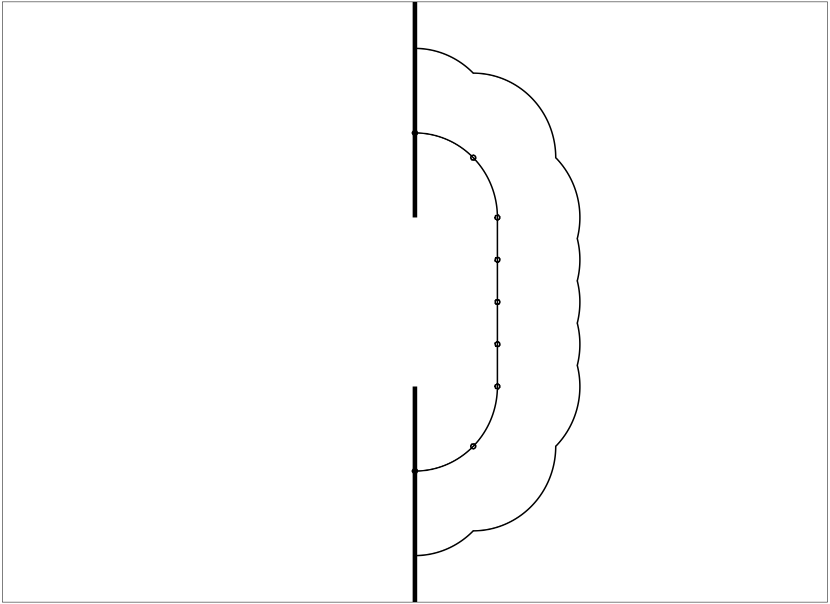
Wavefront propagation via Huygen-Fresnel construction

Leading edge of all the wavelets



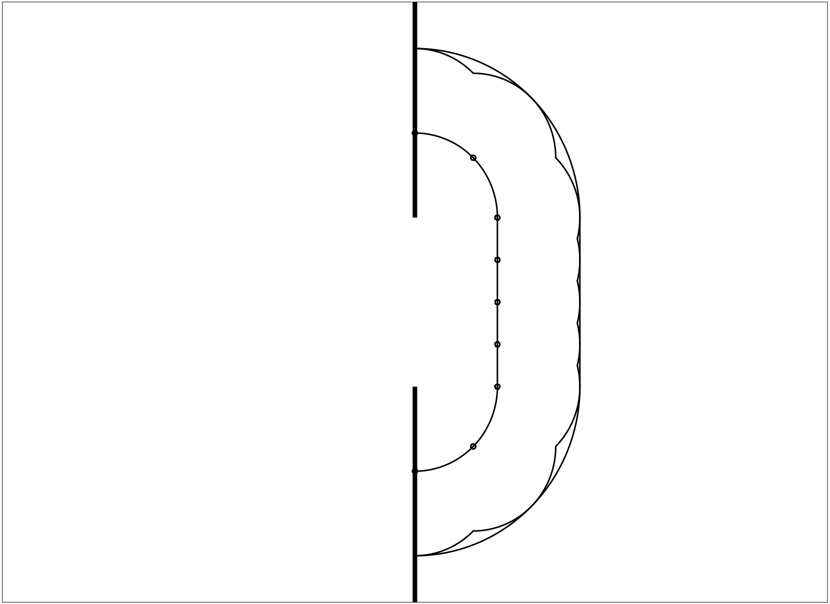
Wavefront propagation via Huygen-Fresnel construction

Approximate form of new wavefront



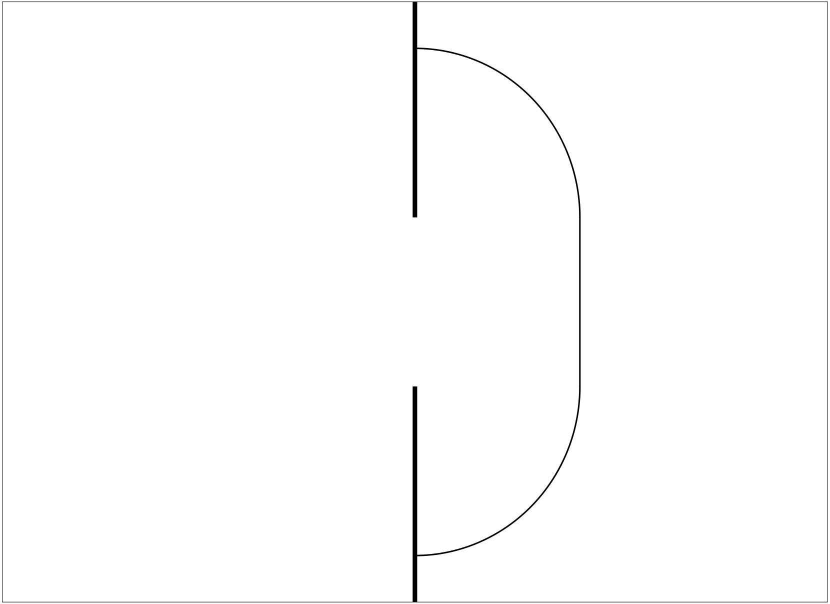
Wavefront propagation via Huygen-Fresnel construction

Fitting the new wavefront



Wavefront propagation via Huygen-Fresnel construction

The next wavefront constructed . . . and so on!

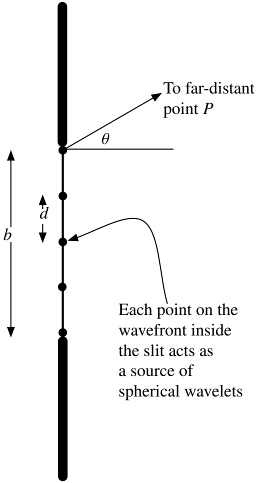


Implementing the Huygen-Fresnel construction

- Replace a wavefront by a collection of secondary sources of Huygen's wavelets.
- Treat each source as being of equal strength and equal phase
- Combine (i.e. add together) the amplitudes of the waves radiated by all of the sources
- Take the limit in which the number of sources is allowed to go to infinity
  - The sources then continuously fill the whole of the wavefront.
  - This usually means that the strength of each source also goes to zero, but in such a way that the total amplitude due to all the waves combined is finite.
- Note that this is an *approximate procedure*, but the Huygen's wavelets idea is basically sound.
- A full analysis of the propagation of waves shows that the amplitude of the wavelets is maximum in the forward direction, and falls to zero in the backward direction.
  - So there are no waves propagating in the backward direction.

Fraunhofer diffraction through a narrow slit

- Shall apply the Huygen-Fresnel construction to analyse the diffraction of waves through a narrow slit in the Fraunhofer limit.



- The distance of the observation point P is $\gg$ width of the slit and $\gg$ the wavelength of the waves.
  - The other extreme is known as Fresnel diffraction, and is much more complex.
- Shall assume there are M sources, all of amplitude a and separated by a distance

$$d = \frac{b}{M-1}$$

- From earlier work, the amplitude at P will be

$$y(P) = a\sin(\omega t - kx + \tfrac{1}{2}(M-1)\delta')\frac{\sin(\frac{1}{2}M\delta')}{\sin(\frac{1}{2}\delta')}$$

where $$\delta' = \frac{2\pi d}{\lambda}\sin\theta = \frac{2\pi b}{(M-1)\lambda}\sin\theta.$$

Fraunhofer diffraction through a narrow slit

The limit of many sources

- From the amplitude $$y(P) = a\sin(\omega t - kx + \tfrac{1}{2}(M-1)\delta')\frac{\sin(\frac{1}{2}M\delta')}{\sin(\frac{1}{2}\delta')}$$

  the time averaged intensity will be $$\bar{I}(P) = \tfrac{1}{2}a^2\frac{\sin^2(\frac{1}{2}M\delta')}{\sin^2(\frac{1}{2}\delta')}$$

  where $$\delta' = \frac{2\pi d}{\lambda}\sin\theta, \qquad d = \frac{b}{M-1}$$

- We want the limit as $M \to \infty$. The sources are then continuous along the wavefront.

- Need to look at the numerator and denominator separately

- $$\begin{aligned}\sin^2(\tfrac{1}{2}M\delta') &= \sin^2\left[\cancel{\tfrac{1}{2}}M\frac{\cancel{2}\pi\sin\theta}{\lambda}\,\frac{b}{M-1}\right]\\ &= \sin^2\left[\frac{\pi b\sin\theta}{\lambda}\cdot\frac{M}{M-1}\right]\\ &\to \sin^2\left[\frac{\pi b\sin\theta}{\lambda}\right] \quad \text{as} \quad M \to \infty.\end{aligned}$$

- $$\begin{aligned}\sin^2(\tfrac{1}{2}\delta') &= \sin^2\left[\frac{\pi d}{\lambda}\sin\theta\right]\\ &= \sin^2\left[\frac{\pi b}{\lambda}\sin\theta\cdot\frac{1}{M-1}\right]\\ &\to \left(\frac{\pi b}{\lambda}\sin\theta\right)^2\cdot\frac{1}{M^2} \quad \text{as} \quad M \to \infty\end{aligned}$$

- Recall, $\sin x \approx x$ if $x \ll 1$.

Fraunhofer diffraction through a narrow slit

Result for continuous line of sources

- In the limit of infinitely many sources we find

$$\bar{I}(P) = \lim_{M\to\infty} \tfrac{1}{2} a^2 M^2 \left(\frac{\sin\left(\frac{\pi b}{\lambda} \sin\theta\right)}{\frac{\pi b}{\lambda} \sin\theta} \right)^2 .$$

- A slight problem:
  - If a is held fixed, then as $M \to \infty$ the expression on the right hand side will diverge!
  - But physically, the result has to be finite.
  - So, the strength of each individual source of Huygen's wavelets must tend to zero as $M \to \infty$. In fact, we must have

$$a \propto \frac{1}{M}.$$

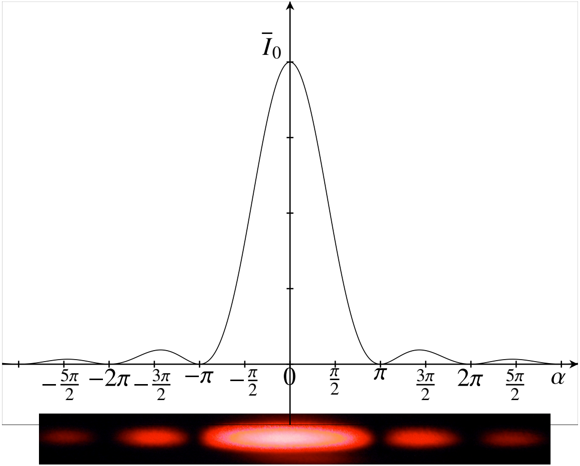
  - So put $\bar{I}_0 = \lim_{M\to\infty} \frac{1}{2} a^2 M^2$. ($\bar{I}_0$ is a constant, but we do not know its meaning yet.)
- Finally,

$$\bar{I}(P) = \bar{I}_0 \frac{\sin^2\alpha}{\alpha^2} \quad \text{with} \quad \alpha = \frac{\pi b}{\lambda} \sin\theta.$$

Single slit diffraction pattern

- The diffraction pattern is given by

$$\bar{I}(\alpha) = \bar{I}_0 \frac{\sin^2 \alpha}{\alpha^2} \quad \text{with} \quad \alpha = \frac{\pi b}{\lambda} \sin \theta.$$



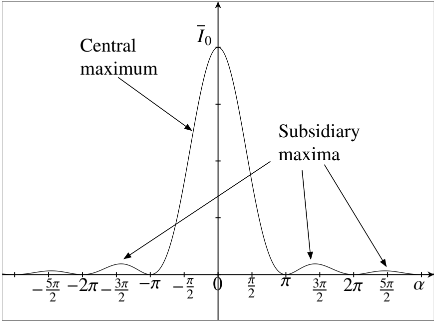
Structure of single slit diffraction pattern

- **Maxima**
  - Can find the positions of the maxima by differentiation:

    $$\frac{d\bar{I}}{d\alpha} = 2\bar{I}_0 \frac{\sin\alpha}{\alpha}\left(\frac{\cos\alpha}{\alpha} - \frac{\sin\alpha}{\alpha^2}\right) = 0$$

    which gives $\tan\alpha = \alpha$.
  - This has one obvious solution: $\alpha = 0$. Since

    $$\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha} = 1 \quad \text{then} \quad \bar{I}(0) = \bar{I}_0$$
  - Thus $\bar{I}_0$ is the intensity of the central $\alpha = 0$ peak of the diffraction pattern.

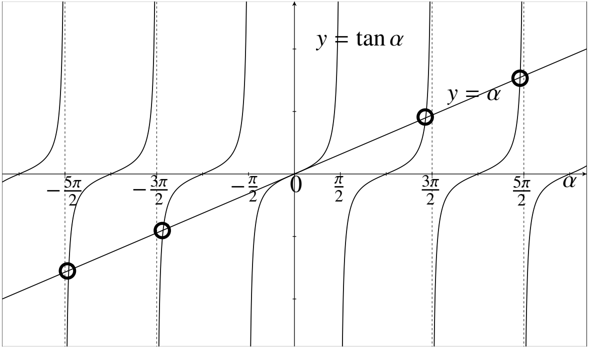


- But there are other maxima to be found by solving $\tan\alpha = \alpha$.
  - This is a *transcendental equation* for which there is no exact solution.
  - Easiest way of solving $\tan\alpha = \alpha$ is *graphically*.

Structure of single slit diffraction pattern

Graphical solution for subsidiary maxima

- Obtain solution by plotting simultaneously $y = \alpha$ and $y = \tan\alpha$.



- The curves intercept at $\alpha = 0$ and approximately at $\alpha \approx \pm(n + \frac{1}{2})\pi$, $n = 1, 2, 3, \ldots$.

- Since $\alpha = \frac{\pi b}{\lambda}\sin\theta$, maxima at

$$b\sin\theta = 0 \qquad \text{and} \qquad b\sin\theta \approx \pm(n + \tfrac{1}{2})\lambda.$$

Structure of single slit diffraction pattern

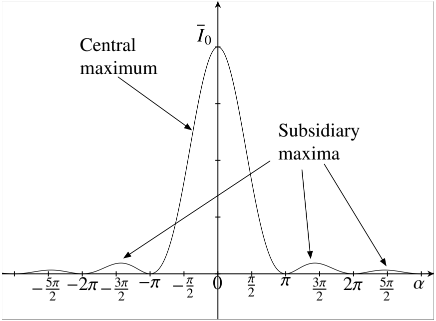
Intensity of maxima.

- The approximate maximum values follow from $\bar{I}(P) = \bar{I}_0 \left(\frac{\sin \alpha}{\alpha} \right)^2$ and are:

$$\bar{I}_{\text{max}} = \bar{I}_0 \quad \text{for} \quad \alpha = 0$$

and

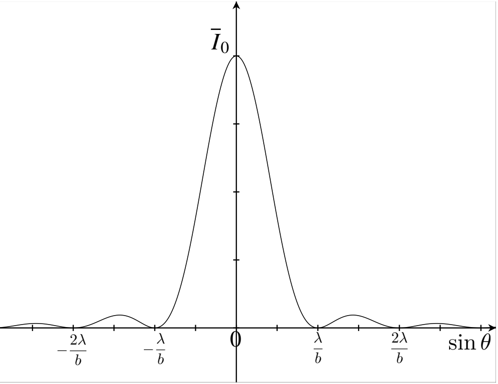
$$\begin{aligned} &\approx \bar{I}_0 \left(\frac{\sin((n + \frac{1}{2})\pi)}{(n + \frac{1}{2})\pi} \right)^2 \quad \text{for} \quad \alpha = (n + \tfrac{1}{2})\pi \\ &= \bar{I}_0 \frac{1}{(n + \frac{1}{2})^2 \pi^2} \\ &= \frac{4\bar{I}_0}{(2n + 1)^2 \pi^2} \end{aligned}$$



- The first subsidiary maximum at $\alpha = \pm 3\pi/2$ has an intensity of $\bar{I}_{\text{max}}(n = 1) = 0.045\bar{I}_0$ so it is 4.5% of the central peak.
- The intensity of subsequent peaks fall off rapidly as $1/n^2$.

Structure of single slit diffraction pattern

- **Minima**



- Minima occur when $\bar{I} = 0$ i.e.

$$\bar{I}_0 \left(\frac{\sin \alpha}{\alpha} \right)^2 = 0$$

$$\Rightarrow \sin \alpha = 0$$

- Exclude $\alpha = 0$ — gives the central maximum.
- So minima occur at

$$\alpha = m\pi, \qquad m = \cancel{0}, \pm 1, \pm 2 \ldots$$

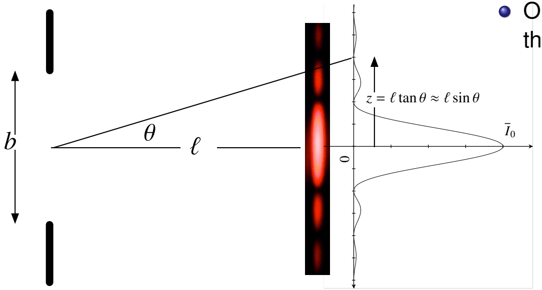
$$\therefore \qquad \frac{\pi b}{\lambda} \sin\theta = m\pi$$

$$\text{or} \qquad b \sin\theta = m\lambda$$

Beware: this looks like $d \sin\theta = n\lambda$, the equation for interference *maxima*.

- The first minima on either side of the central maximum occur at $b \sin\theta = \pm\lambda$
  - Increasing the wavelength or decreasing the slit width makes the central maximum wider

Polar plot and observation screen diffraction pattern



- On **observation screen** a distance ℓ from the slit, have usual approximation

$$z = \ell\tan\theta \approx \ell\sin\theta$$

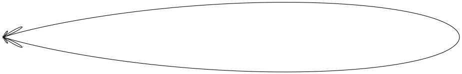
  - So maxima occur at

$$z \approx (n + \tfrac{1}{2})\frac{\ell\lambda}{b} \quad n = \pm1, \pm2, \dots$$

  - Minima at

$$z \approx m\frac{\ell\lambda}{b} \quad m = \pm1, \pm2, \dots$$

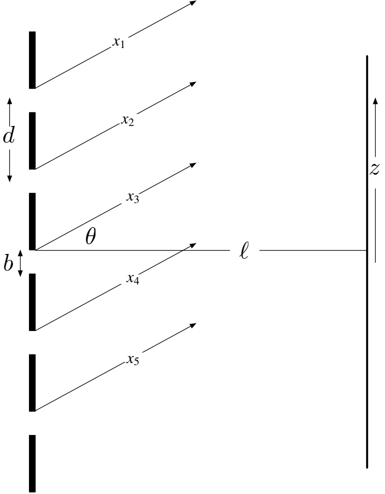
  - Figure on left for $b = 3\lambda$



- A **polar plot** of intensity as a function of angle θ for $b = 3\lambda$.
  - weak subsidiary maxima at $b\sin\theta = \pm(n + \frac{1}{2})\lambda$, $n \neq 0$ i.e. $\sin\theta = \pm\frac{1}{2}, \pm\frac{5}{6}$
  - Minima at $b\sin\theta = m\lambda$, $m \neq 0$ i.e. $\sin\theta = \pm\frac{1}{3}, \pm\frac{2}{3}, \pm1$

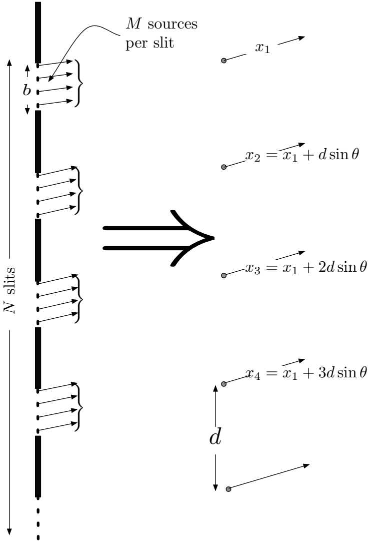
The diffraction grating

- A diffraction grating (or transmission grating) consists of very many narrow, equally spaced, parallel slits.



- Constructed, for instance, by scratching many fine parallel lines on a sheet of glass.
  - The region between the scratches is clear: form the slits of non-zero width
  - The scratches themselves are opaque and determine the distance between slits.
- Light is passed through the grating, producing a pattern on an observation screen which is a combination of:
  - The interference pattern due to the many slits separated by a distance d
  - And the (unavoidable) diffraction pattern associated with the width b of each slit.

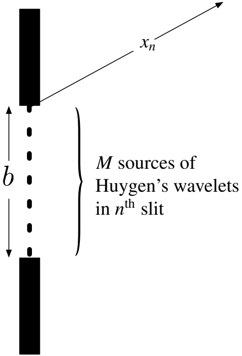
Calculation of diffraction grating interference pattern



- Pattern is calculated in two steps
  - **First** calculate the amplitude of the waves at the observation point produced by one slit of width b
    - This is the Huygen's wavelets single slit diffraction calculation just done.
    - Find that the total amplitude produced by a single slit looks the same as that produced by a single source
    - So the N slits are replaced by N single sources separated by a distance d
  - **Then** we use our much earlier result for the interference pattern of N sources to get the final interference/diffraction pattern.

Calculation of diffraction grating interference pattern I

Contribution of a single slit



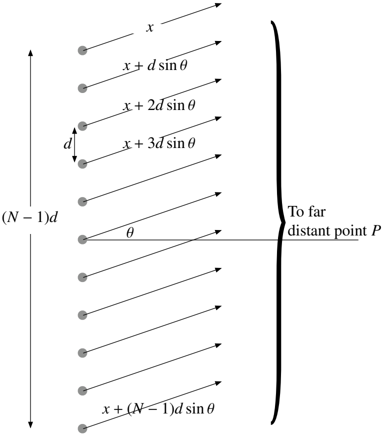
- The total amplitude at observation point P due to waves from the n^{th} slit is, from earlier work:

$$y_n(P) = a\frac{\sin\left(\frac{1}{2}M\delta'\right)}{\sin\left(\frac{1}{2}\delta'\right)}\sin\left(\omega t - kx_n - (M-1)\delta'/2\right)$$

where $$\delta' = \frac{2\pi b}{M-1}\sin\theta.$$

- This is just the formula for a wave produced by a point source
  - of amplitude $a\frac{\sin\left(\frac{1}{2}M\delta'\right)}{\sin\left(\frac{1}{2}\delta'\right)}$
  - phase $(M-1)\delta'/2$
  - and distance x_n from the observation point P.

Calculation of diffraction grating interference pattern II



- Each source has an amplitude $a\dfrac{\sin\left(\frac{1}{2}M\delta'\right)}{\sin\left(\frac{1}{2}\delta'\right)}$
- The n^{th} source is a distance $x_n = x + nd\sin\theta$ from the point of observation P.
    - This is exactly the set-up of N equidistant sources analyzed earlier.
- So the intensity of the waves produced by all the sources is

$$\bar{I}(P) = \frac{1}{2}a^2\left(\frac{\sin\left(\frac{1}{2}M\delta'\right)}{\sin\left(\frac{1}{2}\delta'\right)}\right)^2\left(\frac{\sin\left(\frac{1}{2}N\delta\right)}{\sin\left(\frac{1}{2}\delta\right)}\right)^2$$

- Taking the limit $M \to \infty$ as before then gives

$$\bar{I}(P) = \bar{I}_0\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2 \qquad \alpha = \frac{\pi b}{\lambda}\sin\theta \quad \text{and} \quad \beta = \frac{\pi d}{\lambda}\sin\theta$$

Structure of diffraction pattern for diffraction grating

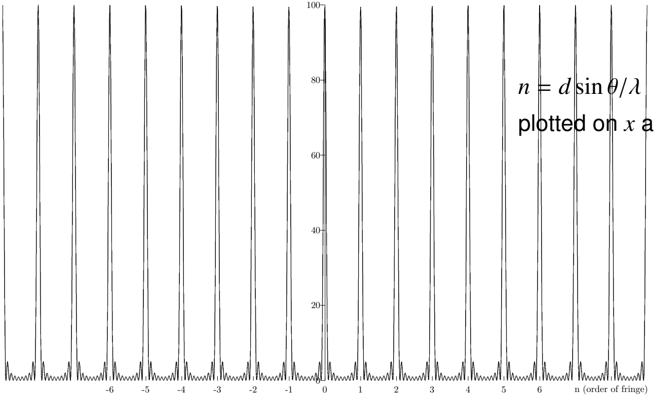
- The Fraunhofer intensity pattern produced by waves of wavelength λ incident on a grating of N slits, all of width b and a distance d apart is

$$\bar{I}(P) = \bar{I}_0 \left(\frac{\sin\alpha}{\alpha}\right)^2 \times \left(\frac{\sin N\beta}{\sin\beta}\right)^2$$

= single slit diffraction pattern × N slit interference pattern

$d = 4b$

$N = 10$



$n = d\sin\theta/\lambda$

plotted on x axis.

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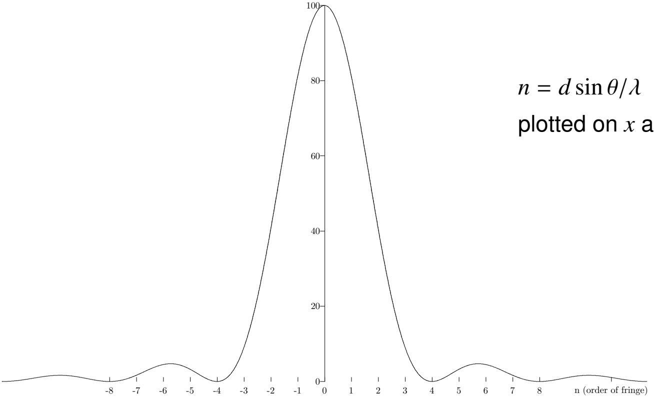

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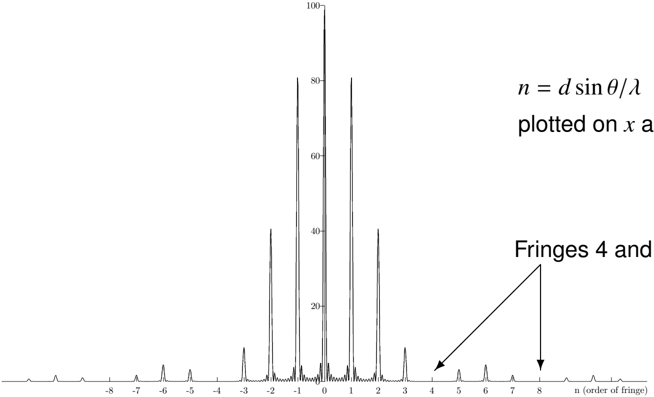

$n = d\sin\theta/\lambda$

plotted on x axis.

Fringes 4 and 8 missing

Missing Fringes

- Recall the following features of interference and diffraction patterns:
  - Maxima of the interference pattern occur when $d \sin\theta = n\lambda$
  - Minima of the diffraction pattern occur when $b \sin\theta = m\lambda$
- If an interference maximum coincides with a diffraction minimum, then the interference maximum will be 'missing'.
  - This will occur if the direction θ is **both** an interference maximum $d \sin\theta = n\lambda$ **and** a diffraction minimum $b \sin\theta = m\lambda$.
  - Combining the two gives $\dfrac{d}{b} = \dfrac{n}{m}$.
- If $d/b = n_0/m_0$ where n_0 and m_0 are integers with no common factors, then
  - any interference fringe $n = rn_0$ where r is an integer will coincide with the diffraction minimum $m = rm_0$
  - Hence every ${n_0}^{\text{th}}$ fringe will be missing.
  - E.g. if $d/b = 4/1$ then every fourth interference fringe will be missing (see previous graphs).
  - If $d/b = 3/2$ then every third fringe will be missing.

Missing fringes continued