Some Useful Properties of Complex Numbers

Complex numbers take the general form $z = x + iy$ where $i = \sqrt{-1}$ and where x and y are both real numbers. There are a few rules associated with the manipulation of complex numbers which are worthwhile being thoroughly familiar with. They are summarized below.

- **Real and imaginary parts** The real and imaginary parts of the complex number $z = x + iy$ are given by

$$\text{Real part} \quad \mathrm{Re}\, z = x \qquad \text{Imaginary part} \quad \mathrm{Im}\, z = y. \tag{1}$$

  with

$$\mathrm{Re}(az_1 + bz_2) = a\mathrm{Re}(z_1) + b\mathrm{Re}(z_2) \text{ and } \mathrm{Im}(az_1 + bz_2) = a\mathrm{Im}(z_1) + b\mathrm{Im}(z_2) \tag{2}$$

  where a and b are both real numbers.

- **Complex conjugate** The complex conjugate of a complex number z, written z^* (or sometimes, in mathematical texts, $\bar{z}$) is obtained by the replacement $i \to -i$, so that $z^* = x - iy$.

- **The modulus of a complex number** The product of a complex number with its complex conjugate is a real, positive number:

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 \tag{3}$$

  and is often written

$$zz^* = |z|^2 = x^2 + y^2 \tag{4}$$

  where

$$|z| = \sqrt{x^2 + y^2} \tag{5}$$

  is known as the *modulus* of z.

- **Euler's theorem** The complex number e^{ix} can be written

$$e^{ix} = \cos x + i \sin x \tag{6}$$

  from which follows:

  (a) $\cos x = \mathrm{Re}\left[e^{ix}\right] \qquad \sin x = \mathrm{Im}\left[e^{ix}\right]$

  (b) The complex conjugate of e^{ix} is e^{-ix} so that

$$e^{-ix} = \cos x - i \sin x. \tag{7}$$

  (c) which leads us to the following important results, the first by adding Eq. (6) and Eq. (7), the second by finding their difference:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{8}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{9}$$

  The last two results are well worth trying to commit to memory.

- **Polar form** A complex number z can be written in the form:

$$z = re^{i\theta}$$

where

$$r = \sqrt{|z|} = \sqrt{x^2 + y^2} \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}.$$

Thus

$$|z|^2 = \left|re^{i\theta}\right|^2 = re^{i\theta}\, re^{-i\theta} = r^2 e^{(i\theta - i\theta)} = r^2 \quad (\text{NOT } r^2 e^{2i\theta}).$$

Application to problems in interference and diffraction

When there are only a small number of sources, the total intensity of the waves produced by all the sources can be calculated by use of simple trigonometric formula. However, when the number of sources becomes large, this can be an exceedingly complicated procedure. However, complex number methods can offer considerable simplification.

We can note that the kind of waves encountered can be expressed in the form

$$y = a\sin(\omega t - kx + \phi) \tag{10}$$

and that, typically, we have to combine or superimpose two or more waves, that is, add them together e.g. for two sources

$$y = a_1 \sin(\omega t - kx_1 + \phi_1) + a_2 \sin(\omega t - kx_2 + \phi_2) \tag{11}$$

and so on – later we will be combining many such terms.

The value of complex nmbers comes from the need, as we shall see, of carrying out a sum of the form

$$S = \sin b + \sin(b-\delta) + \sin(b-2\delta) + \sin(b-3\delta) + \ldots + \sin\left[b - (N-1)\delta\right] \tag{12}$$

i.e. a sum of N rather similar trignometric terms.

As it stands, this is quite a tricky sum to carry out, but we can turn it into something much simpler by writing

$$\sin(b - n\delta) = \mathrm{Im}\left[e^{i(b-n\delta)}\right] \tag{13}$$

so that

$$\begin{aligned} S =& \mathrm{Im}\left[e^{ib} + e^{i(b-\delta)} + e^{i(b-2\delta)} + \ldots + e^{b-i((N-1)\delta)}\right] \\ =& \mathrm{Im}\left[e^{ib}\left\{1 + e^{-i\delta} + e^{-2i\delta} + \ldots + e^{-i(N-1)\delta}\right\}\right]. \end{aligned} \tag{14}$$

Here we have used the fact that $\mathrm{Im}\left[z_1 + z_2\right] = \mathrm{Im}[z_1] + \mathrm{Im}[z_2]$.

If we put $r = e^{-i\delta}$, we recognize the series between the curly brackets $\{\ldots\}$ as a geometric series with a common ratio r:

$$\begin{aligned} S =& \mathrm{Im}\Big[e^{ib}\Big\{1 + r + r^2 + r^3 + \ldots r^{N-1}\Big\}\Big] \\ =& \mathrm{Im}\left[e^{ib}\frac{1 - r^N}{1 - r}\right] \\ =& \mathrm{Im}\left[e^{ib}\frac{1 - e^{-iN\delta}}{1 - e^{-i\delta}}\right]. \end{aligned} \tag{15}$$

The next step is to try to make use of the formulae Eq. (8) and Eq. (9) given above for sin and cos. To do this we 'extract' a factor $\exp(-iN\delta/2)$ from the numerator and $\exp(-i\delta/2)$ from the denominator of the fraction appearing in Eq. (15) which produces the result:

$$\begin{aligned} S =& \mathrm{Im}\left[e^{ib}\ \frac{e^{-iN\delta/2}}{e^{-i\delta/2}}\frac{e^{iN\delta/2} - e^{-iN\delta/2}}{e^{i\delta/2} - e^{-i\delta/2}}\right] \\ =& \mathrm{Im}\left[e^{ib}\ e^{-i(N-1)\delta/2}\ \frac{\sin N\delta/2}{\sin\delta/2}\right] \\ =& \frac{\sin N\delta/2}{\sin\delta/2}\mathrm{Im}\Big[e^{i(b-(N-1))\delta/2)}\Big] \end{aligned} \tag{16}$$

from which follows

$$S = \frac{\sin N\delta/2}{\sin\delta/2}\sin\big[b - (N-1)\delta/2\big]. \tag{17}$$

Thus we have shown that

$$\begin{aligned} \sin b + \sin(b-\delta) + \sin(b-2\delta) + \sin(b-3\delta) + \ldots + \sin\big[b - (N-1)\delta\big] \\ = \frac{\sin N\delta/2}{\sin\delta/2}\sin\big[b - (N-1)\delta/2\big] \end{aligned} \tag{18}$$

a very useful result that will be applied to the case of large number of identical sources, and later to the case of diffraction through a single slit.