Fractals
General reading...

http://jfgouyet.fr/fractal/fractauk.html

http://algorithmicbotany.org/papers/

Will follow treatment here

and of course ...
• Lots of early mathematical studies in related areas..
• Really introduced in 1970 by Mandelbrot.
• Many structures in nature have fractal appearance: trees, coastlines, some leaves, mountains, clouds, distribution of galaxies ...
• fractal = “infinitely broken” from (frangere, to break)
Topological dimension

Intuitively, it's the number of numbers needed to specify a point on an object

1 number for a curve

2 numbers for a surface

3 numbers for a volume
Box-counting Dimension

To measure a length, area, volume, etc (M) .. first cover them in boxes of side $\epsilon$

$$M = N(\epsilon)\epsilon^d$$

Peano, Cantor, Caratheodory etc showed that there existed pathological objects for which the above failed
Box-counting Dimension

\[ N(\epsilon) \]  minimum number of n-dim boxes of side \( \epsilon \) to cover shape

If there exists a \( d \) so that \( N(\epsilon) \sim \frac{1}{\epsilon^d} \) as \( \epsilon \to 0 \)

Then the dimension the shape is \( d \)

i.e. dimension is \( d \) iff there is a constant \( k \geq 0 \)

\[ \lim_{\epsilon \to 0} \frac{N(\epsilon)}{1/\epsilon^d} = k \]

http://www.math.sunysb.edu/~scott/Book331/Fractal_Dimension.html
Box-counting Dimension

take log of both sides

\[ \lim_{\epsilon \to 0} (\ln N(\epsilon) + d \ln \epsilon) = \ln k \]

solve for \( d \)

\[ d = \lim_{\epsilon \to 0} \frac{\ln k - \ln N(\epsilon)}{\ln \epsilon} \]

\( \ln(k) \) is constant but divided by something becoming infinite

\[ -\lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \epsilon} = d \]

http://www.math.sunysb.edu/~scott/Book331/Fractal_Dimension.html
Coastline

Say we measure the length $L$ of a coastline ...

$$L(\epsilon) = N(\epsilon)\epsilon$$

hence expect

$$\lim_{\epsilon \to 0} \frac{\ln \frac{L(\epsilon)}{\epsilon}}{\ln \epsilon} = -d$$

i.e

$$\lim_{\epsilon \to 0} \frac{\ln L(\epsilon)}{\ln \epsilon} = 1 - d$$
Coastline
Coastlines

Richardson in 1961 studied the lengths of coastlines and borders with variations in the 'ruler' … found a power-law relationship.

A circle for comparison
Self-similarity

If an object is self-similar, there is some expansion factor $r$ by which one can blow up a small copy and get the whole object again. If there are $N$ such copies to make up the set again the box dimension is

$$d = \frac{\ln N}{\ln r}$$

$$N = r^d$$

\[
\begin{align*}
2 &= 2^1 \\
4 &= 2^2 \\
8 &= 2^3
\end{align*}
\]

figs: http://math.rice.edu/~lanius/fractals/dim.html
Sierpinski Triangle

Self similarity in action
What is the box-dimension of the Sierpinski Triangle?

\[ N = r^d \]
What is the box-dimension of the Sierpinski Triangle?

\[ N = r^d \]

\[ 3 = 2^d \]

\[ d = \frac{\ln 3}{\ln 2} \approx 1.585 \]
As a total aside, I have found that methodically drawing the Sierpinski triangle during boring lectures greatly relieves stress. If more boring lectures are anticipated, draw a huge one, like one that spans an entire sheet of regular paper drawn to painstaking detail. After a few lectures your boredom will be greatly relieved, your stress will go down, chicks (or hunks, as the case may be) will dig you, and you'll end up with a really, really impressively detailed (and large) Sierpinski Triangle which people will be really impressed with. They will say things like "Man, that's cool!" and "Whoa, how'dja do that!" and "Man, you must have been really bored."

**Length**

*e.g. von Koch snowflake*

Length increases by factor 4/3 at each iteration

Final length will be infinite

http://local.wasp.uwa.edu.au/~pbourke/fractals/fracintro/
Cantor Set
Koch Curve

Figure 2.10. The construction of the Koch curve.
Koch Snowflake

Regular fractals and self-similarity

Figure 2.14. The Koch island and its construction.
Koch Snowflake
Sierpinski Carpet
Menger Sponge

Figure 2.17. Constructing the Menger sponge.
Mandelbrot set

A set of points on the complex plane whose boundary forms a fractal. Specifically it's the set for which

\[ z_{n+1} = z_n^2 + c \]

remains bounded \((z_0 = 0)\)

eg \(c=1 \rightarrow \{0, 1, 2, 5, 26, \ldots\}\)
(not in set)

eg \(c=i \rightarrow \{0, i, (i-1), -i, (1-i), -i \ldots\}\)
(in set)
Fractals in Nature

http://classes.yale.edu/fractals/Panorama/Nature/NatFracGallery/NatFracGallery.html
e.g. Rivers

e.g. part of Mississippi river

<table>
<thead>
<tr>
<th>side length r</th>
<th>number of boxes N(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>52</td>
</tr>
<tr>
<td>1/8</td>
<td>115</td>
</tr>
<tr>
<td>1/16</td>
<td>275</td>
</tr>
</tbody>
</table>

Log(N(s))

\[ d = 1.2 \]

http://classes.yale.edu/fractals/Panorama/Nature/Rivers/RiverBC.html
e.g. Clouds

\[ D \sim 1.33 \]

**Fig. 2.2.13.** Analysis of the relationship between the area of clouds (varying as \( \lambda^2 \) where \( \lambda \) is a characteristic length ranging from the order of 1 km to several hundred km) and their perimeter. (Lovejoy, 1982).
Remember all the power laws?

fractal structures!
Fig. 2.2.9. Example of fractal construction using affine fractional Brownian functions: Constructions simulating reliefs in 3D space (Mandelbrot, 1982).
L-systems

• L-systems were introduced in 1968 by Lindenmayer as a framework for studying multicellular organisms

• uses a set of string rewriting rules

• see: The Algorithmic Beauty of Plants

http://algorithmicbotany.org/papers/
e.g. Fibonacci numbers

variables: A B
constants: none
start: A
rules: (A → B), (B → AB)

n = 0: A
n = 1: B
n = 2: AB
n = 3: BAB
n = 4: ABBAB
n = 5: BABABBAB
n = 6: ABBABBABABBAB
n = 7: BABABBABABBABBABABBAB

length of each string: 1 1 2 3 5 8 13 21 34 55 89 …

http://en.wikipedia.org/wiki/L-system
e.g. Cantor Dust

variables : A B
constants : none
start : A \{starting character string\}
rules : (A \rightarrow ABA), (B \rightarrow BBB)

Now attach drawing operations to symbols:

A means “draw forward” and B means “move forward”

http://en.wikipedia.org/wiki/L-system
e.g. Sierpinski triangle

variables: A B
constants: + -
start: A
rules: (A → B-A-B), (B → A+B+A)

A&B= “draw forward”
+= “turn left by 60°”
-= “turn right by 60°”
(angle changes sign at each iteration)

http://en.wikipedia.org/wiki/L-system
[The Algorithmic Beauty of Plants]
[The Algorithmic Beauty of Plants]
L-system

http://en.wikipedia.org/wiki/L-system