In previous courses, light has been treated in terms of rays and wavefronts. This can only take us so far since it doesn’t truly account for the wave nature or the polarisation of light. In this course, we treat light as an electromagnetic wave using a more formal approach based on Maxwell’s Equations.

1 Review of wave equation and boundaries

1.1 Maxwell’s Equations and Media (P³ 4-4—4.8, Griffiths 7.3)

To consider propagation of light within dielectric media (eg lenses, prisms etc.), and consider propagation from one media to another media, we need to consider the form of Maxwell’s equations within such media. This will allow us to derive familiar rules such as Snell’s Law as well as many less familiar results.

We have met Maxwell’s Equations in integral and differential forms (PHYS202, PHYS301):

\[ \oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q_{\text{encl}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho \, dV \] \hspace{1cm} \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \hspace{1cm} \text{Gauss' Law (Electric field lines come from charges)}

\[ \oint_S \mathbf{B} \cdot d\mathbf{l} = 0 \] \hspace{1cm} \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \hspace{1cm} \text{(no magnetic monopoles, or magnetic field lines don't end)}

\[ \oint_S \mathbf{B} \cdot d\mathbf{l} = \mu_0 I + \mu_0 \varepsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot dS \] \hspace{1cm} \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \hspace{1cm} \text{Ampère's Law}

(Remember from 202&301, that the surface integrals are read a the integral over the surface, of the local normal component of the field:

\[ \oint_S \mathbf{E} \cdot d\mathbf{n} \, da \]

In free space where \( \rho \) and \( \mathbf{J} \) are zero we have:

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \hspace{1cm} \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \]
with the nice symmetry that is now familiar to us. (Note that \( \mu_0, \varepsilon_0 \) are just constants whose values depend on the definition of our unit system).

The symmetry of these equations is broken if you include the charge (\( \rho \)) term in Gauss’s law and the current density (\( J \)) term in Ampere’s law. Apparently, nature does not have the magnetic equivalent of charge (magnetic monopoles) so we can’t use these to restore complete symmetry.

**Extension:**
Give physical explanations for Maxwell’s Equations. For example, what does \( \oint_S \mathbf{B} \cdot d\mathbf{s} \) mean both mathematically and physically? Why should it equal zero?

- Check for yourself that \( c = (\mu_0 \varepsilon_0)^{-1/2} \) using the known values of \( c, \mu_0 \) and \( \varepsilon_0 \).

### 1.2 Wave Equations in empty space

Take the curl of \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \), and use \( \nabla \times \nabla \times = \text{grad div} - \nabla^2 \) (note \( \nabla \cdot \mathbf{E} = 0 \))

\[
\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.
\]

This gives us a second order differential wave equation:

\[
\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}
\]

where the speed of the wave is \( c = 1 / \sqrt{\mu_0 \varepsilon_0} = (\mu_0 \varepsilon_0)^{-1/2} \).

The parameters \( \mu_0 \) and \( \varepsilon_0 \) and the speed of light had all been measured by the time Maxwell worked all this out, and he found that the waves which were solutions to his wave equation would travel with the speed of light, ie.

Thus in a flash of inspiration, he realised that light was an electromagnetic wave. (Of course, we’re bending the truth here. Maxwell didn’t have the advantage of vector notation and he certainly didn’t use SI. As we saw in PHYS301, in SI, \( c, \mu_0 \) and \( \varepsilon_0 \) are now all defined quantities, and it the strength of the electromagnetic interaction is hiding in the definition of the metre.)

A similar procedure for \( \mathbf{B} \) yields a wave equation of the same form as (1.3) for \( \mathbf{B} \).

### 1.3 Electromagnetic Plane Waves

Solutions to the wave equation will take the general form of:

\[
\begin{align*}
\mathbf{E} &= \mathbf{E}_0 (x, y) \exp \left[ i (kz - \omega t) \right] \\
\mathbf{B} &= \mathbf{B}_0 (x, y) \exp \left[ i (kz - \omega t) \right]
\end{align*}
\]

where \( z \) is the direction of propagation.
We are going to consider plane wave solutions - what do Maxwell’s Equations say about the
directions of \( \mathbf{B} \) and \( \mathbf{E} \)?

A **plane** wave means that there should be no variation of \( \mathbf{B} \) and \( \mathbf{E} \) in the \( x \) and \( y \) directions:
\[
\frac{\partial \mathbf{E}}{\partial x} = \frac{\partial \mathbf{E}}{\partial y} = 0
\]
(1.5)

but from Gauss’s Law in vacuum, we have:
\[
\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0
\]
(1.6)

So \( \frac{\partial E_z}{\partial z} = 0 \) or \( E_z = \text{constant} \). However, from our assumed solution to the wave equation
we have \( \frac{\partial \mathbf{E}}{\partial z} = -i\omega \mathbf{E} \) and the only compatible solution is \( E_z = 0 \).

Similarly for \( \mathbf{B} \) we find that \( B_z = 0 \).  **Extension—show this.**

Thus both \( \mathbf{E} \) and \( \mathbf{B} \) are vectors in the \( x-y \) plane - ie we have **transverse** waves.

To determine the relative orientation of \( \mathbf{E} \) and \( \mathbf{B} \):

Take \( E_0 = E_{0x} \hat{x} + E_{0y} \hat{y} \) and \( \mathbf{E} = E_0 \exp[i(kz - \omega t)] \) and use \( \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \):

\[
\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0
\]

\[
\begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
E_{0x} e^{i(kz-\omega t)} & E_{0y} e^{i(kz-\omega t)} & 0
\end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t}
\]

\[
(-\hat{x}ikE_{0y} + \hat{y}ikE_{0x}) e^{i(kz-\omega t)} = -(-i\omega) B_0 e^{i(kz-\omega t)}
\]

So we have
\[
\mathbf{B}_0 = \frac{1}{c} \hat{z} \times \mathbf{E}_0
\]
(1.7)

and more generally, for a wave propagating in the direction \( \hat{\kappa} \)
\[
\mathbf{B} = \frac{1}{c} \hat{\kappa} \times \mathbf{E}
\]
(1.8)

This means that \( \mathbf{B} \) is **normal** to both \( \mathbf{E} \) and the direction of propagation, \( \hat{\kappa} \)

Given that we have already shown that \( \mathbf{E} \) and \( \mathbf{B} \) are co-planar,

\[ \mathbf{E}, \mathbf{B} \text{ and } \hat{\kappa} \text{ are mutually orthogonal and form a right handed set.} \]

Since \( \hat{\kappa} \) is a unit vector, it also follows that
\[
|\mathbf{B}| = \frac{1}{c} |\mathbf{E}|
\]
(1.9)
In a linear, isotropic medium (more details later) the velocity \( v = \omega / k \), so we have \(|B| = |E|/v|.

Extension:
Some books describe a plane wave as:  \[ E(r,t)= E_0 \exp[ i (k \cdot r - \omega t)] \]
and some books use: \[ E(r,t)= E_0 \exp[ i (k \cdot r - \alpha t)] \]
and yet others use: \[ E(r,t)= \frac{1}{2} [E_0 \exp[ i(k \cdot r - \alpha t)] + cc.\]
where cc. is the complex conjugate.

Is there any difference between these descriptions - how do you find the actual value of the electric field \( E \) at any time and position from these equations? How do you find the irradiance (W/m\(^2\)) from these equations?

### 1.4 Maxwell’s Equations Inside Media (Griffiths 4.1—4.4.1, 7.3)

#### 1.4.1 Electric Fields Inside Media

Remember that an electric field within a polarisable medium will induce a charge separation in each atom in the medium \( \Theta \rightarrow \Theta (= \pm \text{dipole}) \), ie. it polarises the medium to give a net dipole moment. (The polarisation can also arise from the applied field causing a rotation and alignment of the permanent dipole moments of polar molecules.)

#### Uniform Electric Field

Atoms in a medium, no \( E \) field

![Diagram of atoms in a medium without an electric field](image1)

\[ E = 0 \]

Atoms in an applied uniform \( E \) field

![Diagram of atoms in a medium with an electric field](image2)

\[ E > 0 \]

The net effect of the external field is to induce a polarisation, which at any point can be described by the dipole moment per unit volume \( P \). So for a uniform distribution of dipoles each with dipole moment \( p \) and number density \( n \), we have \( P = np \).

Each dipole produces a field:

![Diagram of dipole field](image3)

If we now reverse the problem, and ask what electric field \( E \) will be produced by a given polarisation \( P \)?
Our study of the potential in PHYS301 has given us the tools to work this out. Recall these three results for the scalar potential:

\[ V_{ap}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \] potential at \( \mathbf{r} \) due to dipole at \( \mathbf{r}' \)

\[ V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\mathbf{a}' \] potential at \( \mathbf{r} \) due to surface charge on \( S \)

\[ V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\tau' \] potential at \( \mathbf{r} \) due to charge in volume \( V \)

where the source point is \( \mathbf{r}' \) and observation point is \( \mathbf{r} \). To find the total potential from a distribution of charge, we sum over all the individual charges with the first of Eqs. (1.10), and since the dipole strength varies slowly on the atomic scale, we convert the sum to an integral to give:

\[
V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{\mathbf{p}_i \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \\
= \frac{1}{4\pi\varepsilon_0} \sum_i \frac{\mathbf{p}_i / \Delta V_i \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \Delta V_i \\
\rightarrow \frac{1}{4\pi\varepsilon_0} \int_V \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\tau' \\
= \frac{1}{4\pi\varepsilon_0} \left[ \int_S \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{P}(\mathbf{r}') \cdot d\mathbf{a}' - \int_V \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\tau' \right] \\
= \frac{1}{4\pi\varepsilon_0} \left[ \int_S \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{P}(\mathbf{r}') \cdot \hat{n} \, d\mathbf{a}' - \int_V \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\tau' \right]
\]

The resulting potential has two terms, one that looks like the potential of a surface charge \( \sigma_b(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \hat{n} \), and one that looks like the potential of a volume charge density \( \rho_v = -\nabla \cdot \mathbf{P} \).

The suffix \( b \) indicates that these are bound charges stuck to atoms—they’re not free to move about the medium.

The surface charge term shows up at the edges of a uniform medium, where the charge isn’t cancelled by neighbouring opposite charges, (unless the orientation of the dipoles is perpendicular to the surface).

**Extension**—what is the significance of the dot product with the surface normal in \( \sigma_b = \mathbf{P} \cdot \hat{n} \)? What would happen if it were not present, eg. if we had \( \sigma_b = |\mathbf{P}| \)? Hint: consider the edge of a region of charge for different directions of the applied field.
The bound charge volume density only shows up if the electric field is non-uniform, or if the medium is non-uniform. In that case, the polarisation will not be uniform, i.e. \( \mathbf{P} \rightarrow \mathbf{P}(x,y,z) \) and we can get a non-vanishing divergence in \( \rho_b \).

\[
\begin{array}{c}
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
+ \\
\end{array}
\]

We have bound charges within the medium, giving rise to a volume charge density which produces an electric field.

The medium may also have **free charges imbedded** within it resulting in a free charge density \( \rho_f \) from charges that we put there, in addition to the induced polarisation charge density \( \rho_b \).

The total (volume) charge density that goes in to Gauss’ law becomes

\[
\rho = \rho_b + \rho_f.
\]

And Gauss’s law now reads:

\[
\varepsilon_0 \nabla \cdot \mathbf{E} = \rho_{\text{total}} = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f.
\]

Note that \( \mathbf{E} \) is now the total field—including that portion generated by the polarisation. Now, while we can specify the free charge \( \rho_f \), we usually don’t know the bound charge until the problem is solved.

However, we can combine the two div terms to give

\[
\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f,
\]

and from this we define \( \mathbf{D} \) as

\[
\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P},
\]

where \( \mathbf{D} \) is the **electric displacement**. Note that the meaning of \( \mathbf{D} \) is a little subtle, as compared to the way that \( \mathbf{P} \) is the dipole moment per unit volume, and \( \mathbf{E} \) determines the force on a charge. First and foremost, \( \mathbf{D} \) is a device which allows us to perform calculations more easily, but it does play an important role in the development of electromagnetic and especially optical theory. (It’s also important to remember that properties of \( \mathbf{E} \), do not necessarily apply to \( \mathbf{D} \). For example, \( \mathbf{D} \) is in general **not** curl-free, and therefore there is no potential for \( \mathbf{D} \).)

Gauss’s law now reads:

\[
\iint \mathbf{D} \cdot d\mathbf{s} = Q_{\text{encl}}, \quad \nabla \cdot \mathbf{D} = \rho_f
\]

where \( Q_{\text{encl}} \) is the total *free* charge enclosed within the volume. Remember, we can control free charge, so finding \( \mathbf{D} \) is easier than finding \( \mathbf{E} \).
To close the system of equations, we need a connection between the polarisation and the electric field. If the applied electric field is not too strong, we can assume a linear response for which the induced polarisation is proportional to the electric field:

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E},$$

where $\chi_e$ is the electric susceptibility (it is dimensionless). We now have

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$
$$= \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E}$$
$$= \varepsilon_0 (1 + \chi_e) \mathbf{E}$$
$$= \varepsilon \mathbf{E}$$

and thus both $\mathbf{P}$ and $\mathbf{D}$ are proportional to $\mathbf{E}$, with

$$\mathbf{D} = \varepsilon \mathbf{E},$$

where $\varepsilon = \varepsilon_0 (1 + \chi_e)$ is the permittivity of the material, (cf. $\varepsilon_0 =$ permittivity of free space). It is also common to define $\varepsilon_r = 1 + \chi_e$ as the relative permittivity of the medium, so $\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$. At optical frequencies, typical dielectrics have values of $\varepsilon_r$ in the range [1,15].

It is important to note that the assumption of linear response is only an approximation and is only valid at modest electric field strengths. Later in the course, we will begin to consider the fascinating field of nonlinear optics, where these assumptions break down. A point of terminology: we commonly speak of “nonlinear dielectrics” and occasionally of “linear dielectrics”, but no medium is always linear. Pushed sufficiently hard, any system will behave nonlinearly, but the threshold electric fields at which this occurs varies markedly from one material to another. Usually “linear dielectric” indicates that the system is being operated in a regime where nonlinear effects may be neglected. Nonlinear dielectric may indicate the opposite regime, but is also used to refer to materials whose nonlinear properties are desirable somehow. What those desirable properties might be, we’ll get to later.

### 1.4.2 Magnetic fields in media

Real materials can have a magnetisation which can be thought of as the field produced by bound currents, as opposed to those currents over which we have control, called free currents. Thus we can do a similar treatment as with the electric field where we considered bound and free charges.

The magnetic analog to the polarization is the magnetization $\mathbf{M}$, defined such that $\mathbf{H} = \mathbf{B} / \mu_0 - \mathbf{M}$. In linear magnetic media, we assume that

$$\mathbf{B} = \mu \mathbf{H}$$

where $\mu = \mu_0 (1 + \chi_m)$, and $\chi_m$ is the magnetic susceptibility. We can also write $\mu = \mu_r \mu_0$, with $\mu_r$ known as the relative permeability. Thus the auxiliary field $\mathbf{H}$ is used in magnetostatics in an analogous way to the electric displacement $\mathbf{D}$ in electrostatics. For most dielectrics, $\chi_m = 0$ and $\mu_r = 1$ and the magnetic properties are indistinguishable from free space. Thus for our purposes, we can consider $\mathbf{H}$ and $\mathbf{B}$ to be identical apart from a constant multiplicative
factor. Note that due to one of those unfortunate historical quirks, \( \mu \) enters the theory “upside-down” as compared to \( \varepsilon \) : it’s \( \mathbf{D} = \varepsilon \mathbf{E} \), but \( \mathbf{B} = \mu \mathbf{H} \).

Within a medium, Maxwell’s equations now become:

\[
\oint_S \mathbf{D} \cdot d\mathbf{s} = \mathcal{Q}_{\text{encl}} \quad \nabla \cdot \mathbf{D} = \rho_f \quad \text{Gauss' Law}
\]

\[
\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \quad \nabla \cdot \mathbf{B} = 0
\]

\[
\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot ds \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's Law}
\]

\[
\oint \mathbf{H} \cdot d\mathbf{l} = I_f + \int \frac{\partial \mathbf{D}}{\partial t} \cdot ds \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \quad \text{Ampere's Law}
\]

Note that the names given to the four fields vary according to era, country of origin and personal preference. Most people agree on electric field for \( \mathbf{E} \), and electric displacement for \( \mathbf{D} \), but opinion varies for the magnetic quantities. Many authors call \( \mathbf{H} \) the magnetic field and \( \mathbf{B} \) the magnetic flux density, rather than calling \( \mathbf{B} \) the magnetic field. Most practising physicists just refer to the symbols by letter, and stay out of the debate.

### 1.5 Wave Equations Inside a Medium (Griffiths 9.3)

Now let’s consider wave equations inside a medium with no free charge and no free current. Maxwell’s equations become:

\[
\nabla \cdot \mathbf{D} = 0 \quad \nabla \cdot \mathbf{B} = 0
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}
\]

but if the medium is linear, then \( \mathbf{D} = \varepsilon \mathbf{E} \) and \( \mathbf{B} = \mu \mathbf{H} \), and if medium is homogeneous, so that \( \varepsilon \) and \( \mu \) are independent of position, the equations reduce to:

\[
\nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{B} = 0
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}
\]

which look just like the free space case except that we have replaced \( \mu_0 \varepsilon_0 \) by \( \mu \varepsilon \).

Obviously, in a medium, we can derive the same wave equations, but with a propagation speed of \( \nu = 1/\sqrt{\varepsilon /\varepsilon_0 \mu /\mu_0} \), and correspondingly \( \nu = c/n \) where \( n \) is the refractive index.

As we’ve said, for most materials \( \mu_r = 1 \), so

\[
n = \sqrt{\varepsilon_r} = \sqrt{1 + \chi_r}.
\]
1.5.1 Boundary Conditions in media (Griffiths 7.3.6)

Now we are equipped to deal with Maxwell’s equations at a boundary in media. We’ve seen these derivations a number of times in 202 & 301, so we won’t belabour the point in class, but the details are here for the interested...

Recall that the issue is that Maxwell’s equations or the wave equation, only apply in homogeneous or smoothly varying regions where we can safely take derivatives. At sharp (or fairly sharp) boundaries, we need connection formulae to join up the solutions in each region.

Consider Gauss’ law, \[ \oint \mathbf{D} \cdot d\mathbf{s} = Q_{\text{enc}} \]

applied to the following boundary:

If the enclosed volume is small enough, then Gauss’ law reduces to

\[ (\mathbf{D}_1 \cdot \mathbf{n} - \mathbf{D}_2 \cdot \mathbf{n})s = \sigma_f s \]

where \( \sigma_f \) is the free surface charge density (which will be zero in our case).

Thus, for zero free surface charge, the components of \( \mathbf{D} \) perpendicular to the surface are continuous across the boundary:

\[ D_{1 \perp} = D_{2 \perp} \]

and if the medium is an isotropic dielectric (see later) we have

\[ \varepsilon_1 E_{1 \perp} = \varepsilon_2 E_{2 \perp} \]

Using the second of Maxwell’s laws, remembering that magnetic monopoles (probably) don’t exist, we find by the same argument that:

\[ B_{1 \perp} = B_{2 \perp} \]

Thus the components of \( \mathbf{B} \) perpendicular to the surface are continuous across the boundary.

Applying Faraday’s law to a very thin loop half embedded in the surface, we find
\[ \mathbf{E}_1 \cdot \mathbf{l} - \mathbf{E}_2 \cdot \mathbf{l} = -\frac{\partial \mathbf{B}}{\partial t} \cdot ds \]

where \( ds = n \ da \) is the directed area of the loop. But for our very thin loop, \( da = 0 \), so we have

\[ \mathbf{E}_{1\parallel} = \mathbf{E}_{2\parallel} , \]

ie. the components of the electric field \( \text{parallel} \) to the interface are continuous across the boundary.

Finally using Ampere’s law and assuming no surface currents on our dielectric, we obtain :

\[ \mathbf{H}_{1\parallel} = \mathbf{H}_{2\parallel} , \]

ie. the components of \( \mathbf{H} \) \( \text{parallel} \) to the interface are continuous across the boundary.

To summarise, the boundary conditions for nonmagnetic dielectrics are

\[ \begin{align*}
D_{1\perp} &= D_{2\perp} , \\
B_{1\perp} &= B_{2\perp} , \\
\mathbf{H}_{1\parallel} &= \mathbf{H}_{2\parallel} , \\
\mathbf{E}_{1\parallel} &= \mathbf{E}_{2\parallel} ,
\end{align*} \quad (1.13) \]

and in terms of \( \mathbf{E} \) and \( \mathbf{B} \) only,

\[ \begin{align*}
\varepsilon_1 E_{1\perp} &= \varepsilon_2 E_{2\perp} , \\
\mathbf{E}_{1\parallel} &= \mathbf{E}_{2\parallel} , \\
\mathbf{B}_{1} &= \mathbf{B}_{2} .
\end{align*} \quad (1.14) \]
2 Fresnel Equations (P&P Chapter 23)

2.1 Reflection and Transmission of a Plane Wave

As a first application of the above boundary conditions, we can rigorously derive the laws of geometrical optics (the laws of reflection and refraction). These have been known for many hundreds of years (apparently close to 1000 years in the Arabic literature,) and we first met them in high school using Huygen’s wavefront constructions. It is comforting that they remain valid in the wave picture.

We begin with a plane wave: \( \mathbf{E} = \mathbf{E}_i e^{i(k \cdot \mathbf{r} - \omega t)} \) incident on an interface between two different media.

Let the interface be the x-y plane (ie. defined as \( z = 0 \)).

We assume (guided by observation), that a reflected and transmitted wave may appear, but make no other assumptions. In particular, we don’t assume anything about the relative orientations of the wavevector \( \mathbf{k} \) and the plane of incidence.

In the boundary plane we have three waves existing simultaneously, \( \mathbf{E} \), \( \mathbf{E}_r \) and \( \mathbf{E}_t \):

\[
\mathbf{E} = \mathbf{E}_i e^{i(k \cdot \mathbf{r} - \omega t)} \\
\mathbf{E}_r = \mathbf{E}_0 e^{i(k_r \cdot \mathbf{r} - \omega t)} \\
\mathbf{E}_t = \mathbf{E}_0 e^{i(k_t \cdot \mathbf{r} - \omega t)}
\]

Now the choice of \( r = 0 \) and \( t = 0 \) is arbitrary, so the relationships between \( \mathbf{E} \), \( \mathbf{E}_r \) and \( \mathbf{E}_t \) should not depend on our choice of \( r \) and \( t \). Thus the \( \exp[i(k \cdot \mathbf{r} - \omega t)] \) parts, or the phases \( (k \cdot \mathbf{r} - \omega t) \), of the three waves should all be equal on the boundary:

\[
k \cdot \mathbf{r} - \omega t = \text{constant} \quad \text{on } \mathbf{r} = 0,
\]

In particular, setting \( r = 0 \), for this to be true at all times we need \( \omega_i = \omega_r = \omega_t \), so all waves have the same frequency.

Extension—what would happen if they didn’t?

Setting \( t = 0 \), we get at \( z = 0 \) (ie. on the boundary),
\[
\mathbf{k} \cdot \mathbf{r} = \mathbf{k}_r \cdot \mathbf{r} = \mathbf{k}_t \cdot \mathbf{r},
\]
from which we find
\[
(\mathbf{k} - \mathbf{k}_r) \cdot \mathbf{r} = (\mathbf{k} - \mathbf{k}_t) \cdot \mathbf{r} = 0.
\]
Now \( \mathbf{r} \) can be any point in the plane of the reflecting surface, so \((\mathbf{k} - \mathbf{k}_r) \) and \((\mathbf{k} - \mathbf{k}_t) \) must be normal to the reflecting surface. Thus the three wavevectors lie in the same plane normal to the reflecting surface.

**Extension**—show this diagrammatically.

Now consider \( \mathbf{k} \cdot \mathbf{r} = \mathbf{k}_r \cdot \mathbf{r} \). It’s conventional to measure angles from the normal to the plane, so this implies \( kr \sin \theta_r = k_r r \sin \theta_r \). Since both waves are in the same medium, \( k = k_r, k = k_t \) and so \( \theta = \theta_r \), giving us the **Law of reflection**:

\[
\text{Angle of incidence} = \text{Angle of reflection}
\]

Using \( \mathbf{k} \cdot \mathbf{r} = \mathbf{k}_r \cdot \mathbf{r} \), we have \( kr \sin \theta = k_r r \sin \theta \). Now however, the wavevectors are in different media and we have \( k = \omega n_1 / c \) and \( k_t = \omega n_2 / c \), so we have Snell’s **Law of Refraction**:

\[
n_1 \sin \theta = n_2 \sin \theta_t.
\]

Don’t worry, we haven’t done all that hard work just to prove these two laws!

A few comments.

- Note that we can restate both laws in the fashion: “The parallel component of wavevector is conserved on reflection and transmission”. We’ll make use of this form later on.
- As mentioned, Snell’s Law has recently been discovered to have been known to Abu Sa’d al-’Ala’ Ibn Sahl (940-1000 C.E.), a mathematician in the Abassid court in Baghdad, who also studied the mathematics of lens design.\(^1\) He and co-workers were interested in the design of “burning mirrors”, described in his 984 book, *On the Burning Instruments*. Three centuries later in the *Book of Optics*, Persian physicist Kamal al-Din al-Farisi was proposing a wave nature of light and a correct explanation of the rainbow.

---

\(^1\) For details see the article by Sameen Ahmed Khan in Optics & Photonics News **18** (10), 22 (2007).
Extension The figure below shows a plane wave incident on a boundary between two media. The wavefronts are related by \( n_1 \lambda_1 = \lambda_0 = n_2 \lambda_2 \). Use this and the fact that the wavefronts must match up along the boundary to derive Snell’s Law.

\[ \theta_i, \lambda_1 \]
\[ n_1, \quad \lambda_1 \]
\[ n_2, \quad \lambda_2 \]

2.2 Quantitative Reflection and Transmission at a Boundary: The Fresnel Equations

At this point, we’ve established the directions of the reflected and refracted (transmitted) beams, but we haven’t determined how much of the incident energy is transferred into each. It turns out that this depends on the polarisation of the incident light. While complicating the mathematics somewhat, this has some very useful consequences.

Let the boundary between the two media be the \( x-y \) plane, and the plane of incidence, reflection and refraction be the \( x-z \) plane. (There is no loss of generality in these choices).

Recall, that we can always decompose a plane wave into two orthogonally polarised plane waves. There are two cases for reflection at a boundary depending on the orientation of the \( \mathbf{E} \) field with respect to the plane of incidence.

For the case where the incident wave \( \mathbf{E} \) is linearly polarised in the \(+ y\) direction, that is the \( \mathbf{E} \) field is perpendicular to the plane of incidence, all of the following terms are used

- Transverse Electric (TE) wave
- \( \perp \) polarisation
- \( s \) (senkrecht) polarisation
- \( \sigma \) polarisation

For the case where the incident wave \( \mathbf{E} \) is linearly polarised in the plane of incidence, (or the \( \mathbf{B} \) field is perpendicular to the plane of incidence), all of the following terms are used

- Transverse Magnetic (TM) wave.
- \( || \) polarisation
- \( p \) (parallel) polarisation
• $\pi$ polarisation

Note that the orientation can be tricky to remember. For TE, the electric field is **perpendicular to the plane of incidence**, but parallel to the interface.

Extension—come up with the most interesting and humorous mnemonic you can for remembering which is s and which is p. (There will be a prize).

Let’s begin with TE polarisation...

### 2.2.1 Reflection and Transmission of TE Plane Waves (P&P 23-1)

Given that $E$, $B$, $k$ are mutually orthogonal (and form a right hand set), we can draw the diagram:

From Eq. (1.13), we know that the components of $E$ and $H$ parallel to a boundary are continuous across the boundary.

Thus, for TE, we have

$$E + E_r = E_t$$

$$B\cos\theta - B_r\cos\theta_r = B_t\cos\theta_t$$

Recall that the magnitudes of the fields satisfy $B = E/v = El(c/n)$, giving

$$n_1E\cos\theta - n_1E_r\cos\theta_r = n_2E_t\cos\theta_t$$

$$= n_2(E + E_r)\cos\theta_t$$

Rearranging this, we define the **reflection coefficient**

$$r \equiv \frac{E_r}{E} = \frac{n_1\cos\theta - n_2\cos\theta_r}{n_1\cos\theta + n_2\cos\theta_t}$$

Where we have used $\theta_r = \theta$. If we instead eliminate $E_r$ using $E_r = E_t - E_i$ in
we obtain the transmission coefficient
\[ t \equiv \frac{E_t}{E} = \frac{2n_1 \cos \theta}{n_1 \cos \theta + n_2 \cos \theta_t}. \]

Thus, for the TE case,
\[ r_{TE} = \frac{n_1 \cos \theta - n_2 \cos \theta_t}{n_1 \cos \theta + n_2 \cos \theta_t}, \]
\[ t_{TE} = \frac{2n_1 \cos \theta}{n_1 \cos \theta + n_2 \cos \theta_t}. \]

Note that as a consequence of \( E + E_r = E_t \), the reflection and transmission coefficients satisfy
\[ t_{TE} = 1 + r_{TE}. \]

To use these formulae, we need both \( \theta \) and \( \theta_t \). We find the latter using Snell’s Law. Some texts, including Pedrotti, use Snell’s Law to rewrite the reflection coefficients purely in terms of the incidence angle \( \theta \). This is entirely a matter of taste.

### 2.2.2 Reflection and Transmission of TM Plane Waves

The argument runs similarly for TM polarisation, but this time we have
\[ E \cos \theta + E_r \cos \theta_t = E_t \cos \theta_t \]
\[ -B + B_t = -B_t \]
Again we use \( B = E/\nu = nE/c \) to get

\[ \begin{array}{c}
\end{array} \]
and solving these we get

\[ r_{TM} = \frac{-n_2 \cos \theta + n_1 \cos \theta_i}{n_2 \cos \theta + n_1 \cos \theta_i} \]

\[ t_{TM} = \frac{2n_1 \cos \theta}{n_2 \cos \theta + n_1 \cos \theta_i} \, . \]

The TM coefficients satisfy

\[ n_2 t_{TM} = n_1 \left( 1 + r_{TM} \right). \]

\[ \text{(2.2)} \]

2.3 Notes on Fresnel Equations

Let’s make some observations about properties of the Fresnel equations...

2.3.1 Sign conventions

The sign of the reflection coefficients depends on our conventions for the orientation of the incoming and outgoing fields. In TM, we could have chosen the B field to always lie along +y, in which case the reflection coefficient would differ by a minus sign. So in working out interference effects etc, it’s important to keep the field orientations in mind. Conventions in other texts may differ as well.

2.3.2 Power coefficients

The reflection coefficient \( r \) is an amplitude reflectivity. The power reflectivity or reflectance is given by

\[ R = |r|^2 = r r^* . \]

(In general, the refractive indices may be complex, so we use the modulus squared). However, the power transmission or transmittance \( T \neq r^2 \) except at normal incidence, because we are moving from one medium to another. Therefore the beam area changes, and the irradiance (power/unit area) changes too. We can find the power transmission for a beam from

\[ T = 1 - R \]

using conservation of energy.

Extension—can you find a different equation for the power transmission, by using the Fresnel coefficient and Snell’s Law?

2.3.3 Simple examples

a) Glass-air interface at normal incidence (\( \theta_i = \theta_t = 0 \)), \( n_1 = 1, n_2 = 1.5 \),

\[ r_N = \left( n_1 - n_2 \right) / \left( n_1 + n_2 \right) = -0.2 \] (minus sign gives a 180° phase change reflecting from an optically denser medium), so \( R_N = r_N^2 = 0.04 \) (=4% reflectivity).

Note for Ge, \( n = 4 \), and \( R_N = 36\% \) per surface.

b) Power reflectivity (reflectance) at normal incidence from air: \( R = \left( 1 - n \right)^2 / \left( 1 + n \right) \)
c) Sanity check, if $n_1 = n_2$ (ie. no interface) then $r = 0, t = 1$ (good!)

2.3.4 The Brewster angle

Setting $r_{TM} = 0$, we find the condition $\tan \theta = n_2 / n_1$. The angle $\theta_p = \tan^{-1}(n_2 / n_1)$ is known as Brewster’s angle. With this condition there is zero reflection for $p$ polarisation. This means that if the incident beam is unpolarised, the reflected beam is purely $s$ polarised. This is the reason that polarising sunglasses are useful.

eg. Air to glass, $n_2 = 1.5$, $\theta_p = 56.3^\circ$

At Brewster’s angle, the reflected and transmitted beams are at right angles to each other, ie $\theta_r + \theta_t = \theta + \theta_i = \pi / 2$. This has a nice interpretation in terms of the following diagrams.

Extension—show that the reflected and transmitted beams are at right angles.

It is common practice to use Brewster angled windows on the glass surfaces in lasers, so that there are no reflection losses for one polarisation.

Note that there is a Brewster’s angle for incidence from either side of the boundary, and the two angles sum to $90^\circ$.

Extension

- Prove this last statement: the sum of the Brewster’s angles from either side of the interface is $90^\circ$.
- Why is there no Brewster angle for TE polarization? Try setting $r_{TE}=0$ and find out “what goes wrong”.

2.3.5 Total internal reflection

We can use Snell’s law to replace $n_2 \cos \theta$ in the Fresnel equations by $\frac{n_1}{n_2} \sqrt{(n_2^2 / n_1^2) - \sin^2 \theta}$ but for $n_1 \sin \theta > n_2$ we have a complex reflectivity.

This can only happen when going from a medium of high index to a medium of lower index, ie. for $n_1 > n_2$, and is known as total internal reflection (TIR). TIR occurs for angles of incidence greater than the critical angle $\sin \theta_c = n_1 / n_2$ and for both polarisations.
Note that the numerator and denominator of the reflection coefficient are complex conjugates and so as expected, we have $R = |r|^2 = 1$.

By surrounding a high index medium with a low one, we can effectively trap light in the high index region. This is the basic mechanism of light guidance in the optical fibre and thus lies at the heart of the information revolution. In silica optical fibres, the index difference between the core and cladding is tiny ($n_{co} = 1.444$, $n_{cl} = 1.440$). The critical angle is thus very close to 90°.

Below, we see that although the energy is perfectly reflected at the interface, it’s not true that there is no optical energy in the low index region.

2.3.6 Phase change on reflection
A negative value for the reflection coefficient indicates a $\pi$ phase change upon reflection. That is, the reflected electric field behaves as if it had been delayed by half a wavelength. The presence of a phase shift depends on the polarisation, and for TM light, whether the angle of incidence is beyond the Brewster angle:

Extension

- Calculate Brewster’s angle for water ($n = 1.334$). A fisherman wearing Polaroid™ sunglasses stands on a river bank with his head approx 3 m above the level of the water in the river. Approximately how far out from the river bank will the reflected glare be the least (and hence the fish easiest to spot assuming there are any)?

- Total internal reflection only occurs on one side of an interface, but a Brewster angle exists for incidence from either side. Why? What is the relation between the Brewster angles from medium 1 to medium 2 and the reverse?
Signed reflection coefficient for external/internal reflection

External reflection (air-to-glass)

Internal reflection (glass-to-air)
Reflectances for internal and external incidence.

Phase changes in TIR regime

In the TIR regime, the phase change upon reflection gets more interesting. Consider the TE reflection coefficient,

\[ r_{TE} = \frac{n_l \cos \theta - n_2 \cos \theta_i}{n_l \cos \theta + n_2 \cos \theta_i} = \cos \theta - n \cos \theta_i \]

Where \( n = n_2 / n_1 < 1 \). Using Snell’s Law, we can write

\[ n \cos \theta_i = n \sqrt{1 - \sin^2 \theta_i} = n \sqrt{1 - (n_l / n_2)^2 \sin^2 \theta} = \sqrt{n^2 - \sin^2 \theta} \]

to find

\[ r_{TE} = \frac{\cos \theta - i \sqrt{\sin^2 \theta - n^2}}{\cos \theta + i \sqrt{\sin^2 \theta - n^2}} = e^{-2\alpha} \]

with \( \tan \alpha = \frac{\sqrt{\sin^2 \theta - n^2}}{\cos \theta} \). The total phase change on reflection is thus

\[ \phi_{TE} = -2\alpha = -2 \tan^{-1}\left(\frac{\sqrt{\sin^2 \theta - n^2}}{\cos \theta}\right) \]

The exact formula is not so interesting. What does matter is that the phase change for total internal reflection of TM light is different:
\[ \phi_{TM} = -2 \tan^{-1}\left( \frac{\sqrt{\sin^2 \theta - n^2}}{n^2 \cos \theta} \right) + \pi. \]

There are thus a number of phase change regimes, that depend on polarisation, external or internal incidence, and angle of incidence compared to the critical angle and Brewster’s angle. The results are summarised in these graphs:

**Example**

Find the phase shift for internal and external reflection of both polarisations for a glass \((n=1.5)\) - air interface, at incidence angles of 30 ° and 60°.

**Fresnel rhomb**

In the TIR regime, the difference in phase change between TE and TM is tunable by adjusting the angle of incidence or the materials. This allows to make useful devices:

Suppose we want to convert linearly polarised light to circularly polarised light. We can do this by taking light with equal components of \(s\) and \(p\) and delaying one component by \(\pi/2\). Plotting the difference in phase changes for glass we find a 45° phase difference when \(\theta = 53^\circ\). A Fresnel rhomb is designed to allow two such reflections before the light exits. This provides us with a *quarter wave plate* but using isotropic materials.
2.3.7 Energy distribution in TIR—Frustrated and Attenuated Total Internal Reflection

As mentioned above, while there is no energy transport into the low index medium beyond the critical angle (the time-averaged Poynting vector into the medium vanishes), there is still optical energy present there in the form of an evanescent wave. We see this as follows.

The transmitted wave is given by

$$ E_t = E_{0t} \exp \left[ i \left( k_i \cdot \mathbf{r} - \omega t \right) \right] $$

$$ = i E_{0t} \exp \left[ i \left( k_i \cdot \mathbf{r} - \omega t \right) \right] $$

From the earlier geometry,

$$ k_i \cdot \mathbf{r} = k_i \left( -x \sin \theta_i, 0, -z \cos \theta_i \right), $$

where \( k_i = n_i \omega / c \). From Snell’s Law, \( \sin \theta_i = \sin \theta / n \), and

\( k_i \cos \theta_i = i k_i \sqrt{\sin^2 \theta / n^2 - 1} = i \beta \), so that the transmitted field has the form

$$ E_t = i E_{0t} e^{i(-k_i \sin \theta_i - z \cos \theta_i)} e^{-i\omega t} $$

$$ = i E_{0t} e^{i\left( -x(n_i \omega / c) / (n_i / n_i \sin \theta_i - z \beta) \right)} e^{-i\omega t} $$

$$ = i E_{0t} e^{-i\left[ x \sin \theta_i \omega / c - x \omega / c \right]} e^{-i\omega t}. $$

So the electric field has a propagating character along the \( x \) axis, and a decaying exponential character along the negative \( z \)-axis into the second medium.

It is interesting to examine the energy flux as described by the Poynting vector. After considerable effort, (try it if you like!), this can be shown to be

$$ \mathbf{S}(\mathbf{r}, t) = \varepsilon_0 c |E_t|^2 \left\{ -\hat{x} \sin \theta / n \cos^2 \left( k \sin \theta x - \omega t \right) + \hat{z} \sqrt{\sin^2 \theta / n^2 - 1} \sin \left[ 2 \left( k \sin \theta x - \omega t \right) \right] / 2 \right\} $$

Observe that when we time-average, the energy flow along the \( x \) direction survives but that along \( z \) vanishes. So the wave exists inside the low index medium, but the average energy flow is along the boundary, rather than across it.

This has a number of interesting applications for laboratory devices, particularly for measuring the real and imaginary parts of the refractive index. In Frustrated Total Internal Reflection, we introduce an additional high index medium in the region of the evanescent wave. The wave can then tunnel across the low index medium and escape into the new one.
This process is exactly analogous to quantum mechanical tunnelling through a potential barrier (the mathematics is identical). By measuring how much energy is transported into the third medium we can measure the refractive index difference between the materials.

FTIR has also recently found application in some cool multi-touch screen technologies. See \texttt{www.cs.nyu.edu/~jhan} for details.

Attenuated Total Internal Reflection is a similar idea except that the new medium is lossy which leads to a reduction in the energy in the reflected beam. Measuring reduction allows the imaginary part of the index of the new medium to be determined.

The exponential decay of the field into the low index medium is reminiscent of the exponential damping of waves in lossy media which we saw in PHYS301. (Recall that an EM wave in a conductor could only penetrate a small distance called the skin depth.) In fact, these are very different phenomena.

- In an evanescent wave, energy is localised near the surface but there is no damping—the energy sloshes around like a standing wave.
- In a lossy medium, energy is removed from the wave and dissipated by doing work on the free carriers (typically electrons). Any energy that enters the medium as light is doomed to become heat.

### 2.5 Negative Index Materials and Left Handed Light

We have seen that the refractive index can be written:

\[
   n = \frac{c}{v_r} = \sqrt{\varepsilon \mu_r} = \frac{\varepsilon \mu}{\varepsilon_0 \mu_0}
\]

Now it is quite common for \( \varepsilon \) to be negative (when the optical frequency is greater than an absorption resonance frequency, or in metals when the optical frequency is greater than the plasma frequency). Negative \( \mu \) is also possible in resonant magnetic systems.

In an obscure 1968 paper, the Soviet physicist Veselago considered systems in which both \( \varepsilon \) and \( \mu \) were negative and showed that this would imply a negative refractive index, with strange consequences we’ll examine below. However, since a material with both \( \varepsilon \) and \( \mu \)
simultaneously negative had never been discovered, this remained a theoretical curiosity. The reason that nature apparently doesn’t provide these materials is easy to understand. The negative permittivity and permeability are found near resonances. Resonances for permittivity tend to be associated with individual atomic and molecular electron states, and these occur in the ultraviolet and infrared. But magnetism is associated with collective motion of many electrons, and so these resonances occur at much lower frequencies, in the microwave and terahertz regimes. So the two types of resonances just don’t overlap in natural structures.

Then in the late 1990s, Sir John Pendry (then just John Pendry!), showed that by cleverly engineering structures on a scale much smaller than the wavelength, new resonances could be introduced at virtually any frequency and so artificial materials could exhibit negative $\varepsilon$ and $\mu$. As a result, the topic of electromagnetic metamaterials has exploded in the last decade. The cloaking devices we examined in PHYS301 are examples of these kinds of systems.

A very early example of such a material is shown below (for operation in the microwave). (Image from Andrew Houck, Photonics Spectra). Observe that it consists of a series of split ring resonators and vertical stripes. The former induce a negative magnetic response and the latter a negative electric response.

Let’s explore the consequences of negative materials.

Given $n = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}}$, if only one of $\varepsilon$ or $\mu$ is negative, then $n$ becomes complex. This means that $e^{i(kz-\omega t)}$ includes a real exponential term, which corresponds to a non propagating wave. So such materials are reflective and absorbing, and not terribly interesting.

In fact, to preserve causality (the physicist’s single most treasured principle), it turns out that both $\varepsilon$ and $\mu$ must be complex with positive imaginary parts. Now suppose their real parts are both negative. Then $n \approx (\varepsilon, \mu, + i(\varepsilon, \mu, + \varepsilon, \mu, ))^{1/2}$. Now, if $\text{Im}[n] < 0$, then waves would grow which is impossible without a source of gain. We can then quickly show from the Argand diagram that $\text{Re}[n] < 0$. 


Restating, the energy requirement that the imaginary part of $n$ is positive, leads to the conclusion that if both $\varepsilon$ and $\mu$ have negative real parts, the real part of $n$ must also be negative.

One of Veselago’s curious observations was that such materials obey Snell’s law, and therefore transmit on the same side of the normal to the boundary.

From $n_1 \sin \theta_1 = n_2 \sin \theta_2$, if $n_1$ is 1 (air) then $\theta_2$ becomes negative for positive $\theta_1$.

Such materials can be used to make a perfect lens, i.e. a lens for which there is no resolution limit, regardless of the wavelength. (This fact will be more surprising by the end of the course).

Complete the picture below to see how even a rectangular slab of metamaterial can form an image of an object.
Experimental demonstrations of negative index materials in the optical domain were first published in August 2008 by the group of Xiang Zhang at UC, Berkeley, (with simultaneous papers in both *Nature* and *Science*). Images from J. Valentine et al, “Three-dimensional optical metamaterial with a negative refractive index”, *Nature* 07247, (2008).

Figure 1 | Diagram and SEM image of fabricated fishnet structure.
- a, Diagram of the 21-layer fishnet structure with a unit cell of $p = 860$ nm, $a = 565$ nm and $b = 265$ nm.
- b, SEM image of the 21-layer fishnet structure with the side etched, showing the cross-section. The structure consists of alternating layers of 30 nm silver (Ag) and 50 nm magnesium fluoride (MgF$_2$), and the dimensions of the structure correspond to the diagram in a. The inset shows a cross-section of the pattern taken at a 45° angle. The sidewall angle is 45° and was found to have a minor effect on the transmittance curve according to simulation.
Extension

Recently so-called left-handed materials with negative refractive index have been fabricated. Much of the excitement relates to the ability of a plate of such material to form a perfect lens.

Draw some rays which obey Snell’s law to convince yourself that the block of left-handed material with $n = -1$ shown below will form an image of the object.

Further reading: How might materials be constructed such that they have negative $\mu$ and negative $\varepsilon$?

Wavefronts propagating from +ve $n$ to –ve $n$ to +ve $n$:

http://www.cmpgroup.ameslab.gov/portal/LHM.jpg/view